## TRACE METHODS FOR COHOCHSCHILD HOMOLOGY

SARAH KLANDERMAN AND MAXIMILIEN PÉROUX

ABSTRACT. We provide traces for coHochschild homology, dualizing the Hattori-Stallings trace. Considering the trace of the identity, we define traces between coHochschild homology and certain K-theories of coalgebras. We employ bicategorical methods of Ponto to show that coHochschild homology is a shadow. Consequently, we obtain that coHochschild homology is Morita-Takeuchi invariant.

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#### 1. INTRODUCTION

**Background.** The trace of a square matrix is a fundamental invariant in linear algebra. One of its characteristic properties is that it is additive and cyclic:

 $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B), \qquad \operatorname{tr}(AB) = \operatorname{tr}(BA),$ 

which make the trace easily computable and tractable. The trace also records valuable information on fixed points. For instance, given a finite CW-complex X and a continuous map  $f: X \to X$ , the alternating sum of the traces of f induced on rational homology of X is the Lefschetz number, L(f). If  $L(f) \neq 0$ , then f is has at least one fixed point [Lef26, Lef37].

The definition of the trace can be extended from matrices to any endomorphism on a dualizable object in a symmetric monoidal category [DP80, PS14]. A more subtle situation arises in non-symmetric contexts. For instance, the relative tensor product of bimodules over a non-commutative ring R does not define a symmetric monoidal structure. Nevertheless, in [Hat65, Sta65], the trace of an R-linear endomorphism on a finitely generated projective R-module M is defined as a homomorphism  $\operatorname{End}_R(M) \to R/[R, R]$ , called the Hattori-Stallings trace. Ponto generalized this approach and introduced the notion of traces in *shadowed bicategories* [Pon10]. Essentially, a bicategory is a generalization of a monoidal category in which we allow a change of base ring, and a shadow is a choice of a symmetry in this monoidal structure.

Algebraic K-theory is a crucial invariant for rings. Computations in K-theory can provide new insights in other areas of mathematics; for instance, determining the K-theory of  $\mathbb{Z}$  would resolve the Vandiver conjecture [Kur92], an important problem in algebraic number theory. However, just as with homotopy groups of spheres, K-theory is extremely difficult to compute. Hochschild homology, denoted HH, is another

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invariant for rings which approximates K-theory via the Dennis trace  $K_*(R) \to \text{HH}_*(R)$ , and is easier to compute. At level zero, this is precisely the Hattori-Stallings rank  $K_0(R) \to R/[R, R]$ , i.e., the trace of the identity, since  $\text{HH}_0(R) = R/[R, R]$ . As shown by Bökstedt, calculations of Hochschild homology are simplified by considering its homotopy coherent analogue, called topological Hochschild homology (THH). The approximation can be further refined using cyclotomic structures [NS18]. Most sources of computations of K-theory stem from calculations of HH, THH, and cyclotomic refinements. Algebraic K-theory is the natural home for additive invariants [BGT13], while topological Hochschild homology is the natural home for traces and fixed point invariants [CLM<sup>+</sup>20], which makes the study of Hochschild homology valuable in its own right.

CoHochschild homology (coHH) is the invariant for coalgebras analogous to Hochschild homology. First introduced by Doi [Doi81], the definition was extended to chain complexes in [HPS09], then generalized to the stable homotopy theory setting [HS21], which in this case is called topological coHochschild homology (coTHH). Work of the second author shows that strictly coassociative and counital coalgebras in any monoidal model category of spectra do not model  $\mathbb{E}_1$ -coalgebras in spectra [PS19, Pér22b]. That is, coalgebras in spectra that are coassociative and counital up to higher homotopy cannot be rigidified to a strictly coassociative and counital coalgebra. The result remains true if we replace spectra by modules over R, where R is any  $\mathbb{E}_{\infty}$ -ring spectrum. Essentially, the only possible (strictly coassociative and counital) coalgebras in spectra in the current monoidal model categories are of the form  $\Sigma_+^{\infty} X$ , with comultiplication induced by the diagonal  $X \to X \times X$ . Consequently, only computations of coTHH( $\Sigma_+^{\infty} X$ ) are considered in [HS21, Kla22]. Therefore topological coHochschild homology requires an  $\infty$ -categorical setting in general. This is the approach of [BP23, BGS22].

Recall that given a connected space X, we obtain an equivalence of spectra  $\text{THH}(\Sigma^{+}_{+}\Omega X) \simeq \Sigma^{+}_{+}\mathcal{L}X$ , where  $\mathcal{L}X$  is the free loop space on X [Lod98, 7.3.11]. Further conditions on X induced by Koszul duality show that we obtain an equivalence of spectra  $\text{coTHH}(\Sigma^{\infty}_{+}X) \simeq \Sigma^{\infty}_{+}\mathcal{L}X$  [HS21, 3.7]. This leads to new computations in string topology [BGS22]. Moreover, together with the Dennis trace, Hess-Shipley obtained a new trace to coTHH [HS21]

$$K(\Sigma^{\infty}_{+}\Omega X) \longrightarrow \text{THH}(\Sigma^{\infty}_{+}\Omega X) \simeq \text{coTHH}(\Sigma^{\infty}_{+}X),$$

which is a reformulation of the Euler characteristic on a simply connected space X [CP19]. Induced by Spanier-Whitehead duality, the Dennis trace also determines a pairing [BP23, 4.4]

$$K(C^{\vee}) \wedge \operatorname{coTHH}(C) \longrightarrow \mathbb{S}_{+}$$

for which C is an  $\mathbb{E}_1$ -coalgebra in spectra with some finiteness conditions, and  $C^{\vee}$  is its Spanier-Whitehead dual, and thus an  $\mathbb{E}_1$ -ring spectrum. However, all the above traces are unsatisfying as they are directly induced by combining the usual Dennis trace with an identification of coTHH with THH, and are not internal to the coHochschild setting. Our paper aims to provide a Hattori-Stallings trace for coHochschild homology internally, without restriction on the coalgebras.

**Results.** Let k be a commutative ring with global dimension zero. The quotient  $R \to R/R[R, R] = HH_0(R)$ is regarded as a universal trace and can be extended to the Hattori-Stallings trace tr:  $End_R(M) \to HH_0(R)$ defined on any finitely generated projective right module M over a ring R. Similarly, Quillen introduce the dual notion of universal cotrace cotr:  $coHH_0(C) \to C$  in [Qui88]. He employs this cotrace as a means to algebraically extend Chern-Weil theory on Connes' cyclic complex of a ring (see more details in Example 2.4 below). Just as Hattori and Stallings, we extend the universal cotrace cotr:  $coHH_0(C) \to C$  as follows.

**Theorem 1.1.** Let C be a k-coalgebra. Let M be a finitely cogenerated injective left C-comodule. Then there exists a universal cotrace on M as a k-linear homomorphism

$$\operatorname{cotr}: {}_{C}\mathsf{End}(M) \otimes_{\Bbbk} \operatorname{coHH}_{0}(C) \longrightarrow \Bbbk$$

that is cyclic in the following sense. Given another finitely cogenerated left C-comodule N and C-colinear homomorphisms  $f: M \to N$  and  $g: N \to M$ , then for all  $c \in \text{coHH}_0(C)$  we have

$$\operatorname{cotr}((f \circ g) \otimes c) = \operatorname{cotr}((g \circ f) \otimes c).$$

In the case of a ring R, considering the trace of the identity defines a homomorphism  $K_0(R) \to \operatorname{HH}_0(R)$ , the Hattori-Stallings rank. This is the level zero of the Dennis trace  $K_*(R) \to \operatorname{HH}_*(R)$ . Considering the cotrace of the identity leads to the following.

**Theorem 1.2.** Let C be a k-coalgebra. Denote by  $K_0(C)$  the group completion on the set of isomorphism classes of finitely cogenerated injective left C-comodules with direct sums. The corank defines a Z-linear homomorphism  $K_0(C) \otimes_{\mathbb{Z}} \operatorname{coHH}_0(C) \longrightarrow \mathbb{K}$ , or in other words a k-linear homomorphism

$$\operatorname{coHH}_0(C) \longrightarrow K_0^{\vee}(C),$$

where  $K_0^{\vee}(C) = \operatorname{Hom}_{\mathbb{Z}}(K_0(C), \Bbbk)$ .

This new K-theory captures coalgebraic structures. The Hattori-Stallings corank relates to the usual rank in the following way.

• If we let C be the trivial k-coalgebra k, then  $\operatorname{coHH}_0(\mathbb{k}) = \mathbb{k}$  and k-comodules are just k-modules. Consequently, the cotrace is simply a k-linear homomorphism

$$\operatorname{End}_{\Bbbk}(M) \longrightarrow \Bbbk$$
,

which is the usual trace on M, a finitely generated and projective k-module. Moreover,  $K_0(\Bbbk)$  equals the usual K-theory of  $\Bbbk$  as a ring, and thus  $K_0(\Bbbk) = \mathbb{Z}$ , and  $K_0^{\vee}(\Bbbk) = \Bbbk$ , and the corank is the identity homomorphism on  $\Bbbk$ .

• Coalgebras that are finitely generated as k-modules are equivalent to algebras that are finitely generated as k-modules. Therefore it is to be expected that the corresponding trace methods are equivalent. Indeed, if C is finitely generated as a k-module, then  $\operatorname{coHH}_0(C) \cong \operatorname{HH}_0(C^*)^*$ . The usual rank on  $C^*$  provides a homomorphism  $K_0(C^*) \to \operatorname{HH}_0(C^*)$ , which by taking duals corresponds to a homomorphism  $\operatorname{coHH}_0(C) \to \operatorname{Hom}_{\mathbb{Z}}(K_0(C^*), \mathbb{k})$ , and recovers our cotrace, and the results in [BP23, 4.4].

One can extend the definition of the Hattori-Stallings trace of an endomorphism to a twisted endomorphism  $M \to M \otimes_R P$  where M is finitely generated projective right R-module and P is an (R, R)-bimodule. It defines a homomorphism  $\operatorname{Hom}_R(M, M \otimes_R P) \to \operatorname{HH}_0(P, R)$  (see Definition 2.3 below). We show that if P = C is a k-coalgebra and M has a right C-comodule structure, it can lift to a trace on coalgebras the following way.

**Theorem 1.3.** Let C be a  $\Bbbk$ -coalgebra. Let M be a right C-comodule that is finitely generated as a  $\Bbbk$ -module. Then there exists a universal C-trace on M as a  $\Bbbk$ -linear homomorphism

$$\operatorname{tr}^C \colon \operatorname{End}_C(M) \longrightarrow \operatorname{coHH}_0(C),$$

that is cyclic in the following sense. Given another right C-comodule N that is finitely generated as a  $\Bbbk$ -module, and C-colinear homomorphisms  $f: M \to N$  and  $g: N \to M$ , we have

$$\operatorname{tr}^C(f \circ g) = \operatorname{tr}^C(g \circ f).$$

In short, the C-trace of a C-colinear homomorphism  $f: M \to M$  will be defined as  $\operatorname{tr}^{C}(f) = \operatorname{tr}(\rho \circ f)$ where  $\rho: M \to M \otimes_{\Bbbk} C$  is the coaction on M. Considering the C-trace of the identity yields a rank on an another K-theory more closely related to relative K-theory described below.

**Theorem 1.4.** Let C be a k-coalgebra. Denote by  $K_0^C(\Bbbk)$  the group completion on the set of isomorphism classes of right C-comodules that are finitely generated as k-modules with direct sums. Considering the C-trace of the identity yields a  $\mathbb{Z}$ -linear homomorphism

$$K_0^C(\Bbbk) \longrightarrow \operatorname{coHH}_0(C).$$

Given a topological space X, our K-theory  $K_0^C(\Bbbk)$  has a similar definition as the Waldhausen K-theory of homotopically finite pointed spaces comodules over  $X_+ = X \coprod \{*\}$ , which was shown to relate to the A-theory of X [HS16, 1.3, 1.4].

Just as the Hattori-Stallings trace over perfect chain complex over R can be defined as the alternating sum of the traces at each level, one can carefully extend the cotrace (Theorem 1.1) and C-trace (Theorem 1.3) above to chain complexes. See more in section 6. One of the defining properties of K-theory and Hochschild homology is invariance under Morita equivalences – rings that have equivalent (possibly derived) categories of left modules. The dual notion for coalgebras and comodules is called *Morita-Takeuchi equivalence*. It has been more challenging to show that coHochschild homology is invariant under Morita-Takeuchi equivalence in the derived context due to the pathological nature of model categories of comodules as we recall below. Nevertheless we obtain the following theorem.

**Theorem 1.5.** Let C and D be homotopically Morita-Takeuchi equivalent simply connected differential graded k-coalgebras. Then we obtain an isomorphism:

$$\operatorname{coHH}(C) \cong \operatorname{coHH}(D).$$

**Methods.** All the theorems above follow from the fact that coHochschild homology is a shadow on bicategories of bicomodules (Theorems 4.5 and 4.8). As mentioned above, a shadowed bicategory is a generalization of a symmetric monoidal category, and thus come equipped with a notion of trace. Morita-Takeuchi invariance becomes a property not of coHH itself, but rather its categorical context. It is the natural notion of an equivalence in these bicategories, so coHH is Morita-Takeuchi invariant simply because it is a bicategorical construct.

Theorems 1.1 and 1.3 are therefore consequences of Theorem 4.5 and Proposition 6.2. We first present them in Definition 2.11 and Definition 2.25. Theorems 1.2 and 1.4 are direct corollaries and are introduced in Definition 2.18 and in Definition 2.28. Lastly, Theorem 1.5 is a consequence of Theorem 4.8 and Proposition 7.2, see Example 7.4.

Philosophically, our approach dualize the arguments that show Hochschild homology is a shadow [Pon10], using an adaptation of the Dennis-Waldhausen Morita argument. However, one must be careful as not every result immediately dualizes. Firstly, obtaining model structures on comodules and coalgebras is challenging [BHK<sup>+</sup>15, HKRS17, GKR20]. Even in the case when they do exist, they may not be well behaved. For instance, given a nice enough monoidal model category, its associated module and algebra categories represent modules and algebras that are associative and unital up to higher homotopy, see for instance [Lur17, 4.1.8.4, 4.3.3.17]. While the dual result for comodule and coalgebras might not be even true [PS19, Pér22b]. However, the second author showed that a model structure on connective comodules over simply connected differential graded k-coalgebras up to quasi-isomorphism represent homotopy coherent connective comodules in Hk-spectra [Pér20].

Secondly, a cornerstone requirement to show that HH is a shadow is the existence of a derived tensor product, modeled by a two-sided bar construction. For (bi)comodules, we should then need a two-sided cobar construction to model a derived *cotensor* product. As we point out in Remark 3.11, topologists and algebraists have a different meaning for the cobar construction, and obstacles emerge in both. The topological definition of cobar is well-behaved homotopically, but usually does not preserve comodule structures, while the algebraic definition of cobar does preserve these structures, but is not invariant under quasiisomorphisms. To circumvent these issues, the second author in [Pér20] restricts to the context of simply connected differential graded coalgebras, which is the setting for Koszul duality, which thus allows us to mimic some of the arguments in algebra. Many challenges are still faced, such as totalization not commuting with the tensor product, which we address in this paper.

It is our hope that our results are the first stepping stones toward trace methods for coTHH. As we just recalled, derived coalgebras and comodules need the language of  $\infty$ -categories rather than model categories. Thus, in stable homotopy theory, coTHH needs a notion of a shadow in an ( $\infty$ , 2)-categorical context rather than a bicategorical context, which has yet to be developed [HR21a]. We also hope that these new *K*theories will share similar universal properties and that the traces we have obtained can be recovered in a more abstract structure as in [BGT13].

**Organization.** We finish this section with notation and definitions that are used throughout this paper. In section 2, we introduce the definitions of the cotrace and trace on coHochschild homology. This section does not require any knowledge of bicategories nor homotopy theory. In order to be a self-contained paper, we will recall all necessary bicategorical definitions in later sections. We introduce the different bicategories of bicomodules in Section 3. Our key result, that coHochschild homology defines a bicategorical shadow, is proved in Section 4. We then explore the notion of duality in bicomodules in Section 5, which has been under-documented in the literature thus far. In Section 6, we provide a bicategorical description of the traces that appear in Section 2 and Theorems 1.1 and 1.3. In Section 7, we show how Morita-Takeuchi equivalences are part of the bicategorical structures. In Appendix A, we carefully detail how to represent the derived cotensor product of bicomodules and explore what fibrant bicomodules are. Finally, in Appendix B, we give an interpretation of coHochschild homology with coefficients in higher categories, as it was previously undone, and show how it relates to classical definitions of coHochschild homology. We left the terminology of  $\infty$ -categories in the appendices.

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**Notation.** We begin by establishing notation that we use throughout this paper.

- (1) The letter k shall always denote a commutative ring with global dimension zero, i.e. a finite product of fields. Every k-module is projective and injective.
- (2) Let  $\mathsf{Mod}_{\Bbbk}$  denote the category of  $\Bbbk$ -modules. We write the relative tensor product  $\otimes_{\Bbbk}$  as  $\otimes$ .
- (3) Given k-modules M and C, and k-linear homomorphisms  $\Delta : C \to C \otimes C$  and  $\rho : M \to M \otimes C$ , we may employ the *Sweedler notation* in coalgebraic contexts:

$$\Delta(c) = \sum_{i=1}^{n} c_{(1)_{i}} \otimes c_{(2)_{i}} =: c_{(1)} \otimes c_{(2)}, \text{ and } \rho(m) = \sum_{i=1}^{n} m_{(0)_{i}} \otimes m_{(1)_{i}} =: m_{(0)} \otimes m_{(1)}.$$

(4) Let  $Ch_{\Bbbk}^{\geq 0}$  be the category of non-negative chain complexes of k-modules (graded homologically). The category is endowed with a symmetric monoidal structure. The tensor product of two chain complexes X and Y is defined by

$$(X \otimes Y)_n = \bigoplus_{i+j=n} X_i \otimes_{\Bbbk} Y_j,$$

with differential given on homogeneous elements by

$$d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy.$$

We denote the tensor simply as  $\otimes$ . The monoidal unit is denoted  $\Bbbk$ , which is the chain complex  $\Bbbk$  concentrated in degree zero.

- (5) We write  $\operatorname{Hom}_{\Bbbk}(-,-)$  for the internal hom for both  $\operatorname{Mod}_{\Bbbk}$  and  $\operatorname{Ch}_{\Bbbk}^{\geq 0}$ . We write  $M^* = \operatorname{Hom}_{\Bbbk}(M, \Bbbk)$ , the linear dual.
- (6) Given any symmetric monoidal category  $(\mathsf{C}, \otimes)$ , we write  $\tau : X \otimes Y \xrightarrow{\cong} Y \otimes X$ , the natural symmetric isomorphism.

**General definitions.** We recall here the categorical definitions of coalgebras and comodules, and their related concepts.

**Definition 1.6.** Let  $(\mathsf{C}, \otimes, \mathbb{I})$  be a symmetric monoidal category. A *coalgebra*  $(C, \Delta, \varepsilon)$  in  $\mathsf{C}$  consists of an object C in  $\mathsf{C}$  together with a *coassociative* comultiplication  $\Delta : C \to C \otimes C$ , such that the following diagram commutes:



and that admits a *counit* morphism  $\varepsilon: C \to \mathbb{I}$  such that we have the following commutative diagram:



The coalgebra is *cocommutative* if the following diagram commutes:



Given a coalgebra  $(C, \Delta, \varepsilon)$ , we can define its opposite coalgebra  $C^{\mathsf{op}}$  to be the coalgebra  $(C, \tau \circ \Delta, \varepsilon)$ . If C is cocommutative, then  $C = C^{\mathsf{op}}$ .

**Definition 1.7.** Let  $(\mathsf{C}, \otimes, \mathbb{I})$  be a symmetric monoidal category, and let  $(C, \Delta, \varepsilon)$  be a coalgebra in  $\mathsf{C}$ . A right *C*-comodule  $(M, \rho)$  is an object *M* in  $\mathsf{C}$  together with a coassociative and counital right coaction morphism  $\rho: M \to M \otimes C$  in  $\mathsf{C}$ , i.e., the following diagrams commute:



We can similarly define a left C-comodule  $(M, \lambda)$ . In fact, notice that a left C-comodule is a right  $C^{op}$ comodule. Given two coalgebras C and D in C, a (C, D)-bicomodule  $(M, \lambda, \rho)$  is a left C-comodule  $(M, \lambda)$ and a right D-comodule  $(M, \rho)$  such that the following diagram commutes:

$$\begin{array}{ccc} M & \stackrel{\rho}{\longrightarrow} & M \otimes D \\ & & & \downarrow \\ \lambda \downarrow & & & \downarrow \lambda \otimes \operatorname{id}_D \\ C \otimes M & \stackrel{\operatorname{id}_C \otimes \rho}{\longrightarrow} & C \otimes M \otimes D. \end{array}$$

A morphism  $(M, \rho_M) \to (N, \rho_N)$  of right C-comodules is a morphism  $f : M \to N$  in C such that the following diagram commutes:

$$\begin{array}{c} M & \xrightarrow{f} & N \\ \rho_M \downarrow & & \downarrow \rho_N \\ M \otimes C & \xrightarrow{f \otimes \mathsf{id}_C} & N \otimes C \end{array}$$

This morphism will be referred as a (right) C-colinear homomorphism. A (C, D)-bicolinear homomorphism is a morphism that is both a left C-colinear map and a right D-colinear homomorphism. The notation  $CoMod_C(C)$  will denote the category of right C-comodules in C. Similarly, we use  $_CCoMod(C)$  to denote the category of left C-comodules in C, and  $_CCoMod_D(C)$  for the category of (C, D)-bicomodules. Notice that we have isomorphisms of categories  $_CCoMod_I(C) \cong _CCoMod(C)$  and  $_ICoMod_C(C) \cong CoMod_C(C)$ .

**Definition 1.8.** We say a coalgebra  $(C, \Delta, \varepsilon)$  is *flat* in a symmetric monoidal category  $(C, \otimes, \mathbb{I})$  if the induced functor  $C \otimes -: C \to C$  preserves equalizers when they exist.

**Remark 1.9.** Given a symmetric monoidal category  $(C, \otimes, \mathbb{I})$ , and coalgebras C and D in C, note that we obtain an isomorphism of categories

$${}_{C}\mathsf{CoMod}_{D}(\mathsf{C}) \cong \mathsf{CoMod}_{C^{\mathsf{op}} \otimes D}(\mathsf{C}).$$

In particular, given a (C, D)-bicomodule  $(M, \lambda, \rho)$ , we obtain a right  $(C^{op} \otimes D)$ -comodule via

$$M \xrightarrow{\rho} M \otimes D \xrightarrow{\lambda \otimes \operatorname{id}_{D}} C \otimes M \otimes D \xrightarrow{\tau \otimes \operatorname{id}_{D}} M \otimes C \otimes D.$$

**Definition 1.10.** Let  $(\mathsf{C}, \otimes, \mathbb{I})$  be a symmetric monoidal category, let C, D, and E be coalgebras in  $\mathsf{C}$ , and let  $(M, \lambda_M, \rho_M)$  be a (D, C)-bicomodule and  $(N, \lambda_N, \rho_N)$  be a (C, E)-bicomodule. Define the *cotensor product*  $M \square_C N$  to be the following equalizer in  $\mathsf{C}$ :

$$M \square_C N \longrightarrow M \otimes N \xrightarrow[id_M \otimes \lambda_N]{\rho_M \otimes id_N} M \otimes C \otimes N$$

If D and E are flat coalgebras, then  $M \square_C N$  is a (D, E)-bicomodule via

$$\begin{array}{cccc} M \square_C N & \longrightarrow & M \otimes N & \xrightarrow{\rho_M \otimes \operatorname{id}_N} & M \otimes C \otimes N. \\ & & & & \downarrow^{\lambda_X \otimes \rho_Y} & & \downarrow^{\lambda_M \otimes \operatorname{id}_C \otimes \rho_N} \\ D \otimes (M \square_C N) \otimes E & \longrightarrow & D \otimes M \otimes N \otimes E & \xrightarrow{\operatorname{id}_D \otimes \rho_M \otimes \operatorname{id}_N \otimes \operatorname{id}_E} & D \otimes M \otimes C \otimes N \otimes E. \end{array}$$

**Definition 1.11.** Given a symmetric monoidal category  $(C, \otimes, \mathbb{I})$ , we say two coalgebras C and D in C are *Morita-Takeuchi equivalent* if the categories  $_CCoMod(C)$  and  $_DCoMod(C)$  are equivalent.

**Definition 1.12.** Given a closed symmetric monoidal category  $(\mathsf{C}, \otimes, \mathbb{I}, [-, -])$ , and a coalgebra C, we can provide an enrichment of  $\mathsf{CoMod}_C(\mathsf{C})$  as follows. Given right C-comodules M and N, define  $\mathsf{Hom}_C(M, N)$  as the equalizer in  $\mathsf{C}$ 

$$\mathsf{Hom}_C(M,N) \longrightarrow [M,N] \Longrightarrow [M,N \otimes C],$$

where the first parallel map is induced by the coaction  $N \to N \otimes C$  while the second is defined by forgetfulcofree adjointness. We denote  $\operatorname{End}_{C}(M) = \operatorname{Hom}_{C}(M, M)$ . We can similarly define  $_{C}\operatorname{Hom}(-, -)$  and  $_{C}\operatorname{End}(-)$ for left C-comodules.

**Definition 1.13.** Let A be an abelian symmetric monoidal category with enough injective objects. Suppose C is a flat coalgebra in A. Then  $_C CoMod(A)$  also has enough injective objects. If M is a right C-comodule in A, then the functor  $M\square_C - : {}_C CoMod(A) \to A$  is left exact between abelian categories with enough injectives and thus we can right derive it. We denote by  $CoTor_C^i(M, -)$  the  $i^{th}$  right derived functor. Similarly, if N is a left C-comodule, we can right derive the functor  $-\square_C N : CoMod_C(A) \to A$  and obtain functors  $CoTor_C^i(-, N)$ . Per usual, we can check that  $CoTor_C^i(M, N)$  is unambiguous.

## 2. The Universal Cotrace

In this section, we introduce the traces appearing in Theorem 1.1 and Theorem 1.3. Because it will be crucial in order to dualize it, we carefully recall the various definitions and approaches of the Hattori-Stallings trace as well.

2.1. Hattori-Stallings trace. We recall [Hat65, Sta65]. Let R be a ring, not necessarily commutative, and let A be an abelian group. Denote the ring of square matrices of size n with coefficients in R by  $\mathcal{M}_n(R)$ . A trace function on R with values in A is a collection of set maps  $T_n : \mathcal{M}_n(R) \to A$  for each  $n \ge 1$ , such that  $T_n(M+N) = T_n(M) + T_n(N)$  and  $T_n(MN) = T_n(NM)$ . Each map  $T_n : \mathcal{M}_n(R) \to A$  is entirely determined by the map  $T_1 : R = \mathcal{M}_1(R) \to A$ , [Sta65, 1.4,1.5] as  $T_n(M) = \sum_{i=1}^n T_1(m_{ii})$ , where  $M = [m_{ij}]$ .

Therefore we can rephrase a *trace* on R with values in A to be a  $\mathbb{Z}$ -linear homomorphism  $T: R \to A$  such that T(rs) = T(sr), for all  $r, s \in R$ , i.e. the following diagram commutes:

$$R \otimes_{\mathbb{Z}} R \xrightarrow[m \circ \tau]{} R \xrightarrow{T} A$$

where m denotes the multiplication on R.

If we let  $[R, R] = \{rs - sr | r, s \in R\}$  be the subgroup of commutators, then we obtain a universal trace tr :  $R \to R/[R, R]$  by the quotient homomorphism, and every trace  $T : R \to A$  is uniquely determined by a  $\mathbb{Z}$ -linear homomorphism  $R/[R, R] \to A$ :

Given M a finitely generated free right R-module, then an R-linear endomorphism  $f: M \to M$  is represented by a square matrix on R, and thus any trace  $T: R \to M$  defines a trace  $T: \operatorname{End}_R(M) \to M$  as  $\operatorname{End}_R(M) \cong \mathcal{M}_n(R)$ , where n is the rank of M. More generally, if M is a finitely generated projective right R-module, then there exists another projective module N such that  $M \oplus N$  is a free R-module. Therefore if we extend an *R*-linear endomorphism  $f: M \to M$  to the endomorphism  $f + 0: M \oplus N \to M \oplus N$ , we again obtain a square matrix, and thus any trace  $T: R \to A$  defines a trace  $T: \mathsf{End}_R(M) \to A$ .

**Definition 2.1.** Let R be a ring, and M be a finitely generated projective right R-module. The Hattori-Stallings trace on M is the trace induced by the universal trace  $\operatorname{tr} : R \to R/[R, R]$  on the endomorphisms of M:

$$\operatorname{tr} : \operatorname{End}_R(M) \longrightarrow R/[R,R].$$

The Hattori-Stallings rank of M is then the trace of the identity endomorphism on M. The abelian group R/[R, R] is isomorphic to  $HH_0(R) = R \otimes_{R \otimes R^{op}} R$ , the zeroth Hochschild homology of R.

The definition above depends on the choice of generators of M. However, the Hattori-Stallings can be defined coordinate-free as follows. Given a finitely generated projective right R-module M, we have an isomorphism

$$M \otimes_R \operatorname{Hom}_R(M, R) \xrightarrow{\cong} \operatorname{Hom}_R(M, M) = \operatorname{End}_R(M),$$

and thus the Hattori-Stallings trace is the composition

$$\operatorname{End}_R(M) \xleftarrow{\cong} M \otimes_R \operatorname{Hom}_R(M, R) \xrightarrow{\operatorname{ev}} R \xrightarrow{\operatorname{tr}} \operatorname{HH}_0(R).$$

Let P(R) be the set of isomorphism classes of finitely generated projective right *R*-modules. It is a commutative monoid with respect to direct sum. Then the Hattori-Stallings rank defines an additive map

$$\operatorname{rk}: \mathsf{P}(R) \longrightarrow \operatorname{HH}_0(R)$$
$$[M] \longrightarrow \operatorname{tr}(\operatorname{id}_M).$$

Let  $K_0(R)$  be the group completion of  $\mathsf{P}(R)$ . Then the additive map factors through a  $\mathbb{Z}$ -linear homomorphism

$$\operatorname{rk}: K_0(R) \longrightarrow \operatorname{HH}_0(R).$$

This map is the zeroth level of the Dennis trace  $K_*(R) \to HH_*(R)$ .

**Remark 2.2.** A key observation of [Pon10, 4.2.2] is the following. We can generalize a trace  $T : R \to A$  to a trace on any (R, R)-bimodule P as a  $\mathbb{Z}$ -linear homomorphism  $T : P \to A$  such that T(rx) = T(xr) for all  $r \in R, x \in P$ . A universal trace is then given by  $P \to \operatorname{HH}_0(R, P) := R \otimes_{R \otimes R^{\circ p}} P$ . If M is a right R-module, then  $\operatorname{Hom}_R(M, R) \otimes_{\mathbb{Z}} M$  is an (R, R)-bimodule. The adjoint of the identity morphism  $\mathbb{Z} \otimes_{\mathbb{Z}} M \cong M \to M$ provides a  $\mathbb{Z}$ -linear homomorphism  $\mathbb{Z} \to \operatorname{End}_R(M)$ . The Hattori-Stallings trace,  $\operatorname{tr}(f)$ , on an endomorphism  $f : M \to M$ , where M is finitely generated projective right R-module, is thus the image of 1 under the composition

$$\mathbb{Z} \longrightarrow M \otimes_R M^{\star} \xrightarrow{f \otimes \mathsf{id}} M \otimes_R M^{\star} \xrightarrow{\cong} \operatorname{HH}_0(R, M^{\star} \otimes_{\mathbb{Z}} M) \xrightarrow{\operatorname{ev}} \operatorname{HH}_0(R),$$

where  $M^* = \text{Hom}_R(M, R)$ . This is a bicategorical trace, see Example 6.3. We shall use this approach to obtain a cotrace in comodules.

**Definition 2.3** ([Pon10, 2.3.4]). Let R be a ring, M a finitely generated projective right R-module, and P an (R, R)-bimodule. The Hattori-Stallings trace of a right R-linear homomorphism  $f : M \to M \otimes_R P$ , referred to as an endomorphism on M twisted by P, is the image of f under the composition

$$\operatorname{Hom}_R(M, M \otimes_R P) \xleftarrow{\cong} (M \otimes_R P) \otimes_R \operatorname{Hom}_R(M, R) \xrightarrow{\operatorname{ev}} P \xrightarrow{\operatorname{tr}} \operatorname{HH}_0(P, R).$$

Just as in Remark 2.2, we can define the twisted trace of f in a similar fashion by [Pon10, 4.2.2]. It is the image of 1 under the composition

$$\Bbbk \longrightarrow M \otimes M^{\star} \xrightarrow{f \otimes \mathsf{id}} M \otimes P \otimes M^{\star} \cong \mathrm{HH}_{0}(R, P \otimes M^{\star} \otimes M) \xrightarrow{\mathrm{ev}} \mathrm{HH}_{0}(R, P).$$

2.2. A universal cotrace. We can dualize the above approach as follows. Throughout the rest of the section, let C be a k-coalgebra, not necessarily cocommutative. Let Q be a (C, C)-bicomodule, with left coaction  $\lambda : Q \to C \otimes Q$  and right coaction  $\rho : Q \to Q \otimes C$ . Let V be a k-module. A k-linear map  $T : V \to Q$  is said to be a *cotrace* if the following diagram commutes:

$$V \xrightarrow{T} Q \xrightarrow{\lambda} C \otimes Q.$$

Define the cocommutator submodule  $\langle \langle Q \rangle \rangle_C$  to be the kernel in k-modules:

$$\langle\langle Q \rangle \rangle_C \xrightarrow{\operatorname{cotr}} Q \xrightarrow{\lambda} C \otimes Q.$$

We obtain a universal cotrace cotr :  $\langle \langle Q \rangle \rangle_C \hookrightarrow Q$  in the following sense. Every cotrace  $T: V \to Q$  is uniquely determined by a k-linear homomorphism  $V \to \langle \langle Q \rangle \rangle_C$ :

The universal cotrace already appeared in [Qui88]. Notice that  $\langle \langle Q \rangle \rangle_C$  is isomorphic to the zeroth co-Hochschild homology  $\operatorname{coHH}_0(Q, C) := Q \square_{C \otimes C^{\operatorname{op}}} C$ . We also denote  $\operatorname{coHH}_0(C) := \operatorname{coHH}_0(C, C) \cong \langle \langle C \rangle \rangle_C$ . We recall the definition of  $\operatorname{coHH}_q(Q, C)$  more generally in Definition 4.3.

**Example 2.4.** Given a k-coalgebra C and a k-algebra L with a k-linear trace  $T : L \to V$ , and cotrace  $T': W \to C$ , we then obtain a new trace

$$T^{T'}$$
: Hom<sub>k</sub>( $C, L$ )  $\rightarrow$  Hom<sub>k</sub>( $W, V$ ),

where  $f: C \to L$  is sent to

$$W \xrightarrow{T'} C \xrightarrow{f} L \xrightarrow{T} V.$$

Here  $\operatorname{Hom}_{\Bbbk}(C, L)$  is given a k-algebra structure via convolution [Swe69]. The idea can also be extended to chain complexes. In particular, one can consider the zeroth coHochschild homology of the bar resolution B of a k-algebra R, which Quillen identified with the cyclic complex of R up to a dimension shift [Qui88]. Considering the universal cotrace induced on B, Quillen provides a trace  $T^{\operatorname{cotr}}$  on the differential graded algebra  $\operatorname{Hom}_{\Bbbk}(B, L)$  just as above. As  $B_1 = R$ , a linear homomorphism  $\rho : R \to L$  defines a 1-cochain over R considered as the algebraic analogue of a connection. So its curvature  $\omega$  is defined as  $\delta \rho + \rho^2$  where  $\delta$  is the differential of  $\operatorname{Hom}_{\Bbbk}(B, L)$ . Just as in Chern-Weil theory, the trace  $T^{\operatorname{cotr}}$  on  $\operatorname{Hom}_{\Bbbk}(B, L)$  defines classes  $T^{\operatorname{cotr}}(\omega^n)$  which correspond to cyclic cocycles. This idea is used to construct Connes's odd cyclic classes, their even analogues, which turn out to be Chern-Simons forms, and the Jaffe-Lesniewski-Osterwalder version of the Chern character of Connes's  $\theta$ -summable Fredholm modules.

2.3. The coendomorphism coalgebra. In order to show how the universal cotrace provides a dual analogue of the Hattori-Stallings trace, we need to know the analogue of finitely generated and projective modules for comodules. We shall make use of the notion of "quasi-finite" comodules, as introduced by Takeuchi in [Tak77b]. More modern reviews, in more general settings, can be found in [BW03] and [Al-02].

**Definition 2.5.** We say a left C-comodule M is *finitely cogenerated* if there is an injective C-colinear homomorphism  $M \hookrightarrow C^{\oplus n}$  for some  $n \ge 0$ .

Let  $_C$  CoMod be the category of left C-comodules. If M is a left C-comodule and V is a k-module, then  $M \otimes V$  is a left C-comodule. This defines a functor:

$$M \otimes -: \mathsf{Mod}_{\Bbbk} \longrightarrow {}_C\mathsf{CoMod}$$
$$V \longmapsto M \otimes V.$$

**Definition 2.6.** We say a left *C*-comodule *M* is *quasi-finite* if the functor  $M \otimes -: \mathsf{Mod}_{\Bbbk} \longrightarrow {}_{C}\mathsf{CoMod}$  is a right adjoint. In this case, we denote its left adjoint by  $h_{C}(M, -): {}_{C}\mathsf{CoMod} \to \mathsf{Mod}_{\Bbbk}$ , and refer to it

as the *cohom* functor. In other words, given any left C-comodule N, the cohom  $h_C(M, N)$  is the universal k-module providing a natural k-linear isomorphism

$$\operatorname{Mod}_{\mathbb{k}}\Big(h_{C}(M,N),V\Big)\cong {}_{C}\operatorname{CoMod}\Big(N,M\otimes V\Big),$$

for any k-module V. In particular, for V = k, we obtain the isomorphism

$$h_C(M,N)^* \cong {}_C\operatorname{\mathsf{Hom}}(N,M)$$

**Example 2.7.** An example of a quasi-finite left C-comodule M is given by cofree comodules  $M = C \otimes V$ , where V is a finitely generated k-module. Since we have

$$\begin{aligned} \mathsf{Mod}_{\Bbbk}\Big(h_C(C\otimes V,N),W\Big) &\cong {}_C\mathsf{CoMod}\Big(N,C\otimes V\otimes W\Big) \\ &\cong \mathsf{Mod}_{\Bbbk}\Big(N,V\otimes W\Big) \\ &\cong \mathsf{Mod}_{\Bbbk}\Big(V^*\otimes N,W\Big), \end{aligned}$$

for any k-module W, we obtain  $h_C(C \otimes V, N) \cong V^* \otimes N$ .

Any finitely cogenerated comodule is quasi-finite. The unit of the adjunction provides a k-linear homomorphism called the coevaluation:

$$\operatorname{coev}: N \longrightarrow M \otimes h_C(M, N).$$

If D is another coalgebra, and N is a (D, C)-bicomodule, then  $h_C(M, N)$  is a right D-comodule, and the map coev is a morphism of (C, D)-bicomodules, see [Tak77b, 1.7, 1.19]. Therefore, for any quasi-finite left C-comodule M, we obtain that  $h_C(M, C)$  is a right C-comodule, and we have a (C, C)-bicomodule homomorphism coev :  $C \to M \otimes h_C(M, C)$ .

**Lemma 2.8.** If M is a left C-comodule, and N is a right C-comodule, then  $M \otimes N$  is a (C, C)-bicomodule and we obtain an isomorphism of k-modules

$$\operatorname{coHH}_0(M \otimes N, C) \cong N \square_C M.$$

*Proof.* This will be a particular case of Theorem 4.5.

Let T be a left C-comodule, M a quasi-finite left C-comodule, and N a (C, C)-bicomodule. Denote by  $\partial : h_C(M, N \square_C T) \to h_C(M, N) \square_C T$  the adjoint to the left C-colinear homomorphism

$$\operatorname{coev}\Box \mathsf{id}: N \Box_C T \longrightarrow \Big( M \otimes h_C(M, N) \Big) \Box_C T \cong M \otimes \Big( h_C(M, N) \Box_C T \Big).$$

It is noted that, in [Tak77b, 1.14], the k-linear homomorphism  $\partial$  above is an isomorphism whenever M is an injective quasi-finite left C-comodule. If we choose N = C and T = M, we obtain an isomorphism of k-modules:

$$h_C(M,M) \cong h_C(M,C \square_C M) \xrightarrow{\partial} h_C(M,C) \square_C M$$

**Definition 2.9.** Let M be a quasi-finite left C-comodule. The coendomorphism coalgebra on M denoted  $e_C(M)$  is the k-module  $h_C(M, M)$  endowed with the comultiplication:

$$h_C(M, M) \longrightarrow h_C(M, M) \otimes h_C(M, M),$$

which is the adjoint of the C-colinear homomorphism:

$$M \xrightarrow{\operatorname{coev}} M \otimes h_C(M, M) \xrightarrow{\operatorname{coev} \otimes \operatorname{id}} M \otimes h_C(M, M) \otimes h_C(M, M).$$

The counit  $h_C(M, M) \to \mathbb{k}$  is the adjoint of the identity of the C-colinear homomorphism  $M \stackrel{\cong}{\to} M \otimes \mathbb{k}$ .

**Example 2.10** ([BW03, 12.20]). Let  $M = C^{\oplus n} = C \otimes \Bbbk^{\oplus n}$ ; it is a finitely cogenerated cofree comodule and is thus quasi-finite and injective. Then the coendomorphism coalgebra on  $C^{\oplus n}$  is the *C*-matrix coalgebra  $\mathcal{M}_n(C) := C \otimes \mathcal{M}_n(\Bbbk)$ , for which the coalgebra structure on  $\mathcal{M}_n(\Bbbk)$  is defined as follows. Given any finitely generated  $\Bbbk$ -module *V*, as it is dualizable, we obtain that  $V^* \otimes V$  is a  $\Bbbk$ -coalgebra via evaluation  $V^* \otimes V \to \Bbbk$ (providing the counit) and coevaluation  $\Bbbk \to V^* \otimes V$  (providing the comultiplication by applying it on

 $V^* \otimes V \otimes \mathbb{k}$ ). If we let  $V = \mathbb{k}^{\oplus n}$ , this provides a coalgebra structure on  $\mathcal{M}_n(\mathbb{k})$ . The isomorphism of coalgebras  $\mathcal{M}_n(C) \cong e_C(C^{\oplus n})$  follows from:

 $C \otimes \mathcal{M}_n(\Bbbk) \cong C \otimes \mathsf{Hom}_{\Bbbk}(\Bbbk^{\oplus n}, \Bbbk^{\oplus n}) \xleftarrow{\cong} C \otimes \Bbbk^{\oplus n} \otimes \Bbbk^{\oplus n} \cong C^{\oplus n} \square_C C^{\oplus n} \xleftarrow{\partial} e_C(C^{\oplus n})$ 

Just as in the case for modules, the categories of left C-comodules and left  $\mathcal{M}_n(C)$ -comodules are equivalent, providing an instance of Morita-Takeuchi equivalence.

2.4. A Hattori-Stallings cotrace. We define a Hattori-Stallings cotrace using a dual approach of Remark 2.2. We shall also see that it is induced by the universal cotrace.

**Definition 2.11.** Let M be a finitely cogenerated injective left C-comodule. Let  $f : M \to M$  be a C-colinear endomorphism. We can define a *Hattori-Stallings cotrace of* f as a k-linear homomorphism  $\operatorname{coHH}_0(C) \to \Bbbk$  defined as follows:

$$\operatorname{coHH}_0(C) \xrightarrow{\langle\langle \operatorname{coev} \rangle\rangle} \operatorname{coHH}_0(M \otimes h_C(M, C), C) \cong h_C(M, C) \square_C M \xrightarrow{\operatorname{id}\square f} h_C(M, C) \square_C M \xleftarrow{\partial}_{\cong} e_C(M) \longrightarrow \Bbbk.$$

This defines a k-linear homomorphism  ${}_{C}\mathsf{End}(M) \longrightarrow \mathsf{Hom}_{\Bbbk}(\mathrm{coHH}_{0}(C),\Bbbk)$  and by considering its adjoint, we obtain the Hattori-Stallings cotrace as a k-linear homomorphism:

 $\operatorname{cotr}: {}_{C}\mathsf{End}(M) \otimes \operatorname{coHH}_{0}(C) \longrightarrow \Bbbk.$ 

**Example 2.12.** If  $C = \Bbbk$ , then  $\operatorname{coHH}_0(\Bbbk) = \Bbbk$  and  $\Bbbk$ -comodules are precisely  $\Bbbk$ -modules. A finitely cogenerated injective left  $\Bbbk$ -comodule M is therefore precisely defined as a finitely generated  $\Bbbk$ -module. In this case  $e_{\Bbbk}(M) = h_{\Bbbk}(M, M) = \operatorname{Hom}_{\Bbbk}(M, M)$ . Given a  $\Bbbk$ -linear endomorphism f, the cotrace of f is the  $\Bbbk$ -linear homomorphism  $\Bbbk \to \Bbbk$  sending 1 to  $\operatorname{tr}(f)$ , the usual Hattori-Stallings trace of f.

**Example 2.13.** Suppose  $M = C^{\oplus n}$  is a finitely cogenerated cofree left *C*-comodule. Denote the comultiplication  $C \to C \otimes C$  by  $c \mapsto c_{(1)} \otimes c_{(2)}$ , using Sweedler notation. Then a *C*-colinear endomorphism  $f : C^{\oplus n} \to C^{\oplus n}$  is determined by a k-linear homomorphism  $C^{\oplus n} \to k^{\oplus n}$ , and thus is determined by a k-linear homomorphism

$$C \longrightarrow \mathsf{Hom}_{\Bbbk}(\Bbbk^{\oplus n}, \Bbbk^{\oplus n}) \cong \mathcal{M}_{n}(\Bbbk)$$
$$c \longmapsto f_{c}.$$

Then the Hattori-Stallings cotrace is the following k-linear homomorphism:

$$\operatorname{cotr}: {}_{C}\mathsf{End}(C^{\oplus n}) \otimes \operatorname{coHH}_{0}(C) \longrightarrow \Bbbk$$
$$f \otimes c \longmapsto \varepsilon(c_{(1)}) \operatorname{tr} \left( f_{c_{(2)}} \right).$$

**Remark 2.14.** The Hattori-Stallings cotrace is also induced by the universal cotrace, by carefully dualizing the approach of Definition 2.1. Recall that for any k-module W we have a natural injective k-linear homomorphism  $W \hookrightarrow W^{**}$ , given by evaluation. Given any cotrace  $T: V \to C$ , we can lift it to a cotrace on  $V \to \mathcal{M}_n(C) = C \otimes \mathcal{M}_n(\Bbbk)$  as follows

$$V \longrightarrow \mathcal{M}_n(C)$$
$$v \longmapsto T(v) \otimes I_n,$$

where  $I_n$  is the identity matrix of size n. We can then postcompose with the injection  $\mathcal{M}_n(C) \hookrightarrow \mathcal{M}_n(C)^{**}$ . Recall that  $\mathcal{M}_n(C)^{**} \cong e_C(C^{\oplus n})^{**} \cong {}_C \mathsf{End}(C^{\oplus n})^*$ . Considering the adjoint of the resulting k-linear homomorphism  $V \to \mathcal{M}_n(C)^{**}$  above leads to a pairing:

$$_C \mathsf{End}(C^{\oplus n}) \otimes V \longrightarrow \Bbbk.$$

Considering the universal cotrace  $T = \text{cotr} : \text{coHH}_0(C) \hookrightarrow C$ , we recover the Hattori-Stallings cotrace on the finitely cogenerated cofree left C-comodule  $C^{\oplus n}$ .

More generally, given any finitely cogenerated injective left C-comodule M, by the C-colinear homomorphism  $M \hookrightarrow C^{\oplus n}$ , there exists a left C-comodule N such that  $M \oplus N \cong C^{\oplus n}$  as left C-comodules. Therefore, given any C-colinear endomorphism on M, we can extend by zero to obtain an endomorphism on  $C^{\oplus n}$ . This defines a k-linear homomorphism

$$_C \mathsf{End}(M) \to {_C}\mathsf{End}(C^{\oplus n}).$$

Therefore, if we postcompose the mapping  $V \to \mathcal{M}_n(C)^{**}$  defined above with the induced map  $\mathcal{M}_n(C)^{**} \cong {}_C \mathsf{End}(C^{\oplus n})^* \to {}_C \mathsf{End}(M)^*$ , we obtain a k-linear homomorphism

$$V \longrightarrow {}_C \mathsf{End}(M)^*,$$

whose adjoint is  ${}_{C}\mathsf{End}(M) \otimes V \to \Bbbk$ . If we choose T as the universal cotrace  $T = \operatorname{cotr} : \operatorname{coHH}_0(C) \hookrightarrow C$ , we recover the Hattori-Stallings cotrace on M as in Definition 2.11. We prefer the approach of Definition 2.11 to the one just described as it is more directly dual from the usual case of modules, while above we have to consider the extra injective homomorphism  $\mathcal{M}_n(C) \hookrightarrow \mathcal{M}_n(C)^{**}$ , which feels less natural at first. Moreover, our approach is coordinate-free.

**Remark 2.15.** Although in Definition 2.1, we have the evaluation  $M \otimes_R \operatorname{Hom}_R(M, R) \to R$  for a right R-module M, we do not seem to have a homomorphism  $C \to h_C(M, C) \square_C M$  for a left C-comodule M, but only a morphism  $C \to M \otimes h_C(M, C)$ . This is why Lemma 2.8 is crucial for Definition 2.11. This is an instance of a shadow property, and we will generalize in the following sections (see Theorem 4.5).

2.5. A K-theory for coalgebras. Just as in the case of rings, considering the cotrace of the identity yields a dual notion of a Hattori-Stallings rank. Dually to  $K_0(R)$ , we need to consider a K-theory of coalgebras.

**Definition 2.16.** Let I(C) be the isomorphism class of finitely cogenerated injective left C-comodules. It is a commutative monoid with respect to direct sum. Denote its group completion by  $K_0(C)$ .

**Definition 2.17.** For any  $c \in \text{coHH}_0(C)$ , we can define a *Hattori-Stallings corank* as an additive homomorphism

$$\operatorname{cork}(-,c): \mathsf{I}(C) \longrightarrow \Bbbk$$
  
 $[M] \longmapsto \operatorname{cotr}(\mathsf{id}_M \otimes c)$ 

This lifts to a  $\mathbb{Z}$ -linear homomorphism  $\operatorname{cork}(-, c) : K_0(C) \longrightarrow \Bbbk$ . As the homomorphism  $\operatorname{cork}(-, [M])$  is  $\mathbb{Z}$ -linear, since  $\operatorname{cotr}(\operatorname{id}_M \otimes -)$  is  $\Bbbk$ -linear, the corank is a  $\mathbb{Z}$ -linear homomorphism

$$K_0(C) \otimes_{\mathbb{Z}} \operatorname{coHH}_0(C) \longrightarrow \Bbbk$$

**Definition 2.18.** Define the co-K-theory of C as the k-module  $K_0^{\vee}(C) := \text{Hom}_{\mathbb{Z}}(K_0(C), \mathbb{K})$ . The adjoint of the corank defines a k-linear homomorphism

$$\operatorname{coHH}_0(C) \longrightarrow K_0^{\vee}(C).$$

We first present how the corank relates to the rank in some particular cases.

**Example 2.19.** If  $C = \Bbbk$ , then k-comodules are simply k-modules, and  $\mathsf{I}(\Bbbk) = \mathsf{P}(\Bbbk)$  (as every k-module is both injective and projective). Therefore  $K_0(C) = K_0(\Bbbk) \cong \mathbb{Z}$ ,  $\operatorname{coHH}_0(\Bbbk) = \Bbbk$ , and the corank is simply the identity on  $\Bbbk$ .

**Example 2.20.** Suppose C is finitely generated as a k-module. Since we obtain an equivalence of symmetric monoidal categories on finitely generated k-modules

$$\mathsf{Mod}_{\Bbbk}^{fg} \cong (\mathsf{Mod}_{\Bbbk}^{fg})^{\mathsf{op}},$$

then we can show that  $I(C) \cong P(C^*)$  as commutative monoids, and thus  $K_0(C) \cong K_0(C^*)$ . Note also that we have an isomorphism

$$\operatorname{coHH}_0(C) \cong \operatorname{HH}_0(C^*)^*$$

Therefore, the Hattori-Stallings rank on  $C^*$  will induce the corank as follows. Consider the free k-linear homomorphism induced on the rank

$$\Bbbk \otimes_{\mathbb{Z}} K_0(C^*) \longrightarrow \operatorname{HH}_0(C^*).$$

Apply the k-linear dual on both side to obtain

$$\operatorname{coHH}_0(C) \longrightarrow \operatorname{Hom}_{\Bbbk}(\Bbbk \otimes_{\mathbb{Z}} K_0(C^*), \Bbbk) \cong \operatorname{Hom}_{\mathbb{Z}}(K_0(C^*), \Bbbk).$$

Thus, taking the adjoint, we re-obtain the corank,  $K_0(C^*) \otimes \operatorname{coHH}_0(C) \to \Bbbk$ . This approach was already noted in [BP23, 4.4] for C any  $\mathbb{E}_1$ -coalgebra spectrum, with some finiteness assumption, over an  $\mathbb{E}_{\infty}$ -ring spectrum. When C is a  $\Bbbk$ -coalgebra with no finiteness assumptions, our approach provides a new trace method, more general than the Hattori-Stallings rank. **Remark 2.21** (Eilenberg Swindle). Just as in the case for rings, we need to restrict to finitely cogenerated comodules, as otherwise  $K_0(C) = 0$ . Indeed, let  $\mathbb{k}^{\oplus \infty}$  be the k-vector space with countably infinite basis. Let  $C^{\oplus \infty} = C \otimes \mathbb{k}^{\oplus \infty}$  be an infinitely cogenerated cofree left *C*-comodule. Then we obtain isomorphisms of left *C*-comodules

$$C^{\oplus\infty} = C \otimes \Bbbk^{\oplus\infty}$$
  

$$\cong C \otimes (\Bbbk^{\oplus n} \oplus \Bbbk^{\oplus n} \oplus \cdots)$$
  

$$\cong (C \otimes \Bbbk^{\oplus n}) \oplus (C \otimes \Bbbk^{\oplus n}) \oplus \cdots$$
  

$$\cong C^{\oplus n} \oplus C^{\oplus n} \oplus \cdots$$

Therefore if  $M \oplus N \cong C^{\oplus n}$ , then we obtain

$$M \oplus C^{\oplus \infty} \cong M \oplus (N \oplus M) \oplus (N \oplus M) \oplus \cdots$$
$$\cong (M \oplus N) \oplus (M \oplus N) \oplus \cdots$$
$$\cong C^{\oplus \infty}.$$

Hence, in  $K_0(C)$  we would get [M] = 0.

**Proposition 2.22.** If C is a cocommutative and irreducible k-coalgebra, then  $K_0(C) \cong \mathbb{Z}$  and  $K_0^{\vee}(C) \cong \mathbb{k}$ .

*Proof.* By [Tak77a, A.2.1, A.2.2], injective C-comodules are all cofree when C is cocommutative and irreducible. Then  $I(C) \cong \mathbb{N}$ , and the result follows.

**Proposition 2.23.** If C is a cocommutative  $\Bbbk$ -coalgebra, then  $K_0(C)$  is a commutative ring and  $K_0^{\vee}(C)$  is a cocommutative  $\Bbbk$ -coalgebra if  $K_0(C)$  is finitely generated as an abelian group.

*Proof.* If C is cocommutative, then the cotensor product forms a symmetric monoidal structure on C-comodules. The cotensor product of injective C-comodules is injective [Doi81, Proposition 1]. The cotensor product of finitely cogenerated C-comodules is finitely cogenerated. Therefore  $K_0(C)$  is a commutative ring via cotensor product of C-comodules.

**Example 2.24.** By construction, our K-theory is Morita-Takeuchi invariant. In particular, for any  $n \ge 1$  and any coalgebra C we obtain  $K_0(\mathcal{M}_n(C)) \cong K_0(C)$ .

2.6. Another trace for coalgebras. Instead of dualizing the trace, we can also lift the trace of a twisted endomorphism as in Definition 2.3 over comodules. Let M be a *right* C-comodule that is finitely generated (and automatically projective) as a k-module, and let  $M^*$  be its usual linear dual. A key observation is that in this case we can give a left C-coaction on  $M^*$  via

$$\lambda: M^* \longrightarrow \mathsf{Hom}_{\Bbbk}(M, C) \cong C \otimes M^*$$
$$\left(M \xrightarrow{\alpha} \Bbbk\right) \longmapsto \left(M \xrightarrow{\rho} M \otimes C \xrightarrow{\alpha \otimes \mathsf{id}} C\right),$$

where  $\rho: M \to M \otimes C$  is the right C-coaction of M. By [BW03, 10.11], we have an isomorphism of k-modules  $M \Box_C M^* \cong \operatorname{Hom}_C(M, M)$ . Notice that the following diagram commutes

$$\begin{array}{ccc} M^* \otimes M & \xrightarrow{\lambda \otimes \mathrm{id}} & \operatorname{Hom}_{\Bbbk}(M, C) \otimes M \\ & & & & \downarrow \\ & & & \downarrow \\ M^* \otimes M \otimes C & \longrightarrow C, \end{array}$$

where the unlabeled maps are defined by evaluating. This defines the C-bicolinear evaluation homomorphism  $\varepsilon: M^* \otimes M \to C$ .

**Definition 2.25.** The Hattori-Stallings C-trace of a right C-colinear homomorphism  $f: M \to M$ , denoted by  $\operatorname{tr}^{C}(f)$ , is the Hattori-Stallings trace of  $\rho \circ f: M \to M \otimes C$ . In other words  $\operatorname{tr}^{C}(f) = \operatorname{tr}(\rho \circ f)$ . More precisely, it can be defined as the image of 1 under the composition

$$\Bbbk \longrightarrow \mathsf{End}_C(M) \cong M \square_C M^* \xrightarrow{f \square \mathsf{id}} M \square_C M^* \cong \mathrm{coHH}_0(M^* \otimes M, C) \xrightarrow{\langle \langle \varepsilon \rangle \rangle} \mathrm{coHH}_0(C).$$

The second isomorphism follows from Lemma 2.8. This is simply restricting the trace of  $\rho \circ f$ , as in Definition 2.3. We restrict  $M \otimes M^* \to \operatorname{HH}_0(C, \Bbbk) = C$  to  $M \square_C M^*$ , and the fact that f is C-colinear guarantees we land in  $\operatorname{coHH}_0(C)$ . The C-trace defines a  $\Bbbk$ -linear homomorphism

$$\operatorname{tr}^{C} : \operatorname{End}_{C}(M) \longrightarrow \operatorname{coHH}_{0}(C).$$

**Example 2.26.** Let  $(e_1, \ldots, e_n)$  be a basis of M, and  $(e_1^*, \ldots, e_n^*)$  be the dual basis of  $M^*$ . Denote the coaction  $\rho: M \to M \otimes C$  by  $\rho(m) = m_{(0)} \otimes m_{(1)}$ . Then the C-trace of a C-colinear endomorphism f on M is explicitly given by

$$\operatorname{tr}^{C}(f) = \sum_{i=1}^{n} e_{i}^{*}(f(e_{i})_{(0)})e_{i(1)} \in \operatorname{coHH}_{0}(C).$$

**Definition 2.27.** Denote by  $\mathsf{P}^{C}(\Bbbk)$  the set of isomorphism classes of right *C*-comodules that are finitely generated (and projective) as  $\Bbbk$ -modules. It is a commutative monoid with respect to direct sum. Denote its group completion by  $K_0^{C}(\Bbbk)$ .

Definition 2.28. The Hattori-Stallings C-rank is an additive homomorphism

$$\operatorname{rk}^{C} : \mathsf{P}^{C}(\Bbbk) \longrightarrow \operatorname{coHH}_{0}(C)$$
$$[(M, \rho)] \longmapsto \operatorname{tr}^{C}(\operatorname{id}_{M}) = \operatorname{tr}(\rho).$$

It lifts to a  $\mathbb{Z}$ -linear homomorphism,  $\operatorname{rk}^C : K_0^C(\Bbbk) \to \operatorname{coHH}_0(C)$ .

**Remark 2.29.** While  $K_0(C)$  captures the coalgebraic structure on C,  $K_0^C(\Bbbk)$  is more closely related to relative K-theory,  $K_0(\Bbbk; C)$ , of dualizable  $\Bbbk$ -modules M together with twisted endomorphisms  $M \to M \otimes C$  as in [DM94]. Indeed,  $K_0^C(\Bbbk)$  makes only the additional assumption that the twisted endomorphism defines a right C-coaction.

**Remark 2.30.** Notice that if C and D are Morita-Takeuchi equivalent, then  $K^C(\Bbbk)$  and  $K^D(\Bbbk)$  might not be isomorphic. This phenomenon can already be observed in algebras. Suppose R and S are  $\Bbbk$ -algebras such that left R-modules and left S-modules are categorically equivalent; it does not imply that left R-modules that are finitely generated as  $\Bbbk$ -modules are equivalent to S-modules that are finitely generated as  $\Bbbk$ -modules. A simple example arises from  $R = \Bbbk[x]$  and  $S = \mathcal{M}_{2\times 2}(\Bbbk[x])$ , and considering  $\Bbbk$  as a  $\Bbbk[x]$ -module by assuming x acts trivially.

### 3. Bicategories of bicomodules

In this section, we introduce the bicategories of bicomodules in Definition 3.5 and Definition 3.13. For convenience, we recall the definition of a bicategory, further details of which may be found in [Pon10, PS13, CP19].

**Definition 3.1.** A *bicategory*  $\mathcal{B}$  consists of following.

- A collection of objects,  $ob(\mathcal{B})$ , called *0-cells*. We write  $C \in \mathcal{B}$  instead of  $C \in ob(\mathcal{B})$ .
- Categories  $\mathcal{B}(C, D)$  for each pair of objects  $C, D \in \mathcal{B}$ . The objects in these categories are referred to as *1-cells* and the morphisms as *2-cells*. The composition is referred to as *vertical composition*.
- Unit functors  $U_C \in \mathcal{B}(C, C)$  for all  $C \in \mathcal{B}$ .
- Horizontal composition functors for  $C, D, E \in \mathcal{B}$

$$\odot: \mathcal{B}(C,D) \times \mathcal{B}(D,E) \to \mathcal{B}(C,E),$$

which are not required to be strictly associative or unital.

• Natural isomorphisms for  $M \in \mathcal{B}(C, D)$ ,  $N \in \mathcal{B}(D, E)$ ,  $P \in \mathcal{B}(E, F)$ , and  $Q \in \mathcal{B}(F, G)$  given  $C, D, E, F, G \in \mathcal{B}$ :

$$a: (M \odot N) \odot P \xrightarrow{\cong} M \odot (N \odot P),$$
$$\ell: U_C \odot M \xrightarrow{\cong} M, \qquad r: M \odot U_D \xrightarrow{\cong} M,$$

which satisfy the triangle identity

$$(M \odot U_D) \odot N \xrightarrow{a} M \odot (U_D \odot N)$$

$$r \odot \mathsf{id}_N \xrightarrow{} M \odot N,$$

and the pentagon identity

$$\begin{array}{cccc} (M \odot N) \odot (P \odot Q) \\ ((M \odot N) \odot P) \odot Q & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & &$$

A bicategory with one object is precisely a monoidal category. The bicategory of categories in which 0-cells are categories, 1-cells are functors, and 2-cells are natural transformations is a classical example. We recall other examples below that will be motivating in our context.

**Example 3.2.** We can define a bicategory whose 0-cells are rings and whose 1- and 2-cells come from the category of (R, S)-bimodules for rings R and S. In this case the horizontal composition is given by the tensor product of bimodules. The derived version of this bicategory has 0-cells which are rings and 1- and 2-cells from the derived category of (R, S)-bimodules, where horizontal composition is given by the derived tensor product  $\otimes^{\mathbb{L}}$ . Recall that if M is a right S-module, and N is a left S-module, then  $M \otimes^{\mathbb{L}}_{S} N$  is equivalent to the two-sided bar construction, Bar(M, S, N).

**Example 3.3.** When the 0-cells are given by ring spectra and the 1- and 2-cells come from the homotopy category of (R, S)-bimodules for ring spectra R and S, then the horizontal composition is given by the derived smash product of spectra. Note that as in the previous example,  $Bar(M, S, N) \simeq M \wedge_S^{\mathbb{L}} N$ , see [EKMM97, IV.7.5], and so we may also consider this horizontal composition as the two-sided bar construction.

In this paper, we will introduce two bicategories formed by bicomodules instead of bimodules. Their horizontal compositions will be given by the (derived) cotensor product.

**Proposition 3.4.** Let  $(C, \otimes, \mathbb{I})$  be a symmetric monoidal category, and let C be a flat coalgebra in C. Then  $(_{C}CoMod_{C}(C), \Box_{C}, C)$  is a monoidal category. Further, it is symmetric monoidal if C is cocommutative.

**Definition 3.5.** Define CoMod to be the bicategory whose 0-cells are coalgebras in  $Mod_k$ , and whose 1-cells, 2-cells, and vertical compositions are given by the category  $_CCoMod_D(Mod_k)$ . The unit  $U_C$  is the (C, C)-bicomodule C, and horizontal composition is given by the cotensor product:

For (C, D)-bicomodule M, (D, E)-bicomodule N, and (E, F)-bicomodule P, the natural isomorphisms

$$a: (M \odot N) \odot P \xrightarrow{\cong} M \odot (N \odot P), \qquad \ell: U_C \odot M \xrightarrow{\cong} M, \qquad r: M \odot U_D \xrightarrow{\cong} M,$$

follow as in [Doi81] from the natural isomorphisms

$$(M \Box_D N) \Box_E P \cong M \Box_D (N \Box_E P), \qquad C \Box_C M \cong M, \qquad M \Box_D D \cong M.$$

More specifically, these maps are given by

$$a: (M\square_D N)\square_E P \xrightarrow{\cong} M\square_D (N\square_E P) \qquad \ell: C\square_C M \xrightarrow{\cong} M \qquad r: M\square_D D \xrightarrow{\cong} M \\ (m \otimes n) \otimes p \longmapsto m \otimes (n \otimes p) \qquad c \otimes m \longmapsto \varepsilon_C(c)m \qquad m \otimes d \longmapsto \varepsilon_D(d)m.$$

We now consider the derived case. We first need a homotopy theory of bicomodules. Let  $M = Ch_{\Bbbk}^{\geq 0}$  be the category of non-negative chain complexes over  $\Bbbk$ . There is a model structure on  $Ch_{\Bbbk}^{\geq 0}$  in which weak equivalences are quasi-isomorphisms, cofibrations are monomorphisms, and fibrations are positive levelwise epimorphisms, see [Hov99]. It is a combinatorial symmetric monoidal model category with its usual tensor product, and a simplicial model category via the Dold-Kan correspondence. In addition, every object is cofibrant and fibrant.

**Proposition 3.6.** Let C and D be dg-coalgebras over k. There exists a simplicial combinatorial model structure on  $_{C}CoMod_{D}(Ch_{k}^{\geq 0})$  in which

- the weak equivalences W are the morphisms of (C, D)-bicomodules that are quasi-isomorphisms in  $Ch_{\mathbb{F}}^{\geq 0}$ ;
- the cofibrations are the morphisms of (C, D)-bicomodules that are monomorphisms in  $Ch_{k}^{\geq 0}$ .

In particular, every object is cofibrant.

Proof. The model structure is left-induced in the sense of [HKRS17] via the forgetful-cofree adjunction

$${}_{C}\mathsf{CoMod}_{D}\left(\mathsf{Ch}_{\Bbbk}^{\geq 0}
ight) \xrightarrow[]{U}{\underset{C\otimes -\otimes D}{\perp}} \mathsf{Ch}_{\Bbbk}^{\geq 0}.$$

Indeed, this follows from [Pér20, 2.12] and [HKRS17, 6.3.7] by Remark 1.9.

The above result is true more generally for unbounded chain complexes over any ring. However, it is not known if this forms an algebraic model for homotopy coherent comodules in this generality.

**Definition 3.7.** A (connective) dg-coalgebra over  $\Bbbk$  is a coalgebra C in  $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ . We say it is simply connected if  $C_0 \cong \Bbbk$  and  $C_1 = 0$ .

When C and D are simply connected, we show the model structure from the previous proposition defines the correct homotopy type, see Theorem A.4. Fibrant objects in  $_{C}CoMod_{D}(Ch_{\mathbb{k}}^{\geq 0})$  are retracts of Postnikov towers, see more details in Appendix A.

In order to define horizontal composition for the bicategory in the derived setting, we now need to derive the cotensor product. However, it is more subtle than the usual tensor product of modules because the cotensor product is neither a left nor a right adjoint in general. Nevertheless, one can right derive the cotensor product using methods of [Pér20, Pér21]. We leave the details in Appendix A, but essentially, we show that the cotensor product of fibrant bicomodules is again fibrant (Proposition A.10) and that the cotensor preserves weak equivalences on fibrant objects (Proposition A.17). Moreover, just as the derived tensor product can be interpreted as a two-sided bar construction, the derived cotensor product can be interpreted as a two-sided cobar construction, as we shall now explain.

**Definition 3.8.** Let *C* and *D* be simply connected dg-coalgebras over  $\Bbbk$ , with comultiplication and counit of *C* given by  $\Delta : C \to C \otimes C$  and  $\varepsilon : C \to \Bbbk$  respectively. Let *M* be a (D, C)-bicomodule in  $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ , and let  $\rho : M \to M \otimes C$  denote its right coaction over *C*. The *two-sided cosimplicial cobar construction*  $\Omega^{\bullet}(M, C, C)$ of *M* is the cosimplicial object in  ${}_{D}\mathsf{CoMod}_{C}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ :

$$M \otimes C \Longrightarrow M \otimes C \otimes C \Longrightarrow \cdots,$$

defined as follows.

- For all  $n \ge 0$ ,  $\Omega^n(M, C, C) = M \otimes C^{\otimes n+1}$ .
- The zeroth coface map  $d^0: \Omega^n(M, C, C) \to \Omega^{n+1}(M, C, C)$  is given by  $d^0 = \rho \otimes \mathsf{id}_{C^{n+1}}$ .
- For  $1 \le i \le n+1$ , the *i*<sup>th</sup> coface map  $d^i : \Omega^n(M, C, C) \to \Omega^{n+1}(M, C, C)$  is given by

$$d^{i} = \mathsf{id}_{M} \otimes \mathsf{id}_{C^{\otimes i-1}} \otimes \Delta \otimes \mathsf{id}_{C^{\otimes n+1-i}}.$$

• For all  $0 \leq j \leq n$ , the  $j^{th}$  codegeneracy map  $s^j : \Omega^{n+1}(M, C, C) \to \Omega^n(M, C, C)$  is given by

$$s^j = \mathsf{id}_M \otimes \mathsf{id}_{C^{\otimes j}} \otimes \varepsilon \otimes \mathsf{id}_{C^{\otimes n+1-j}}$$

Since  ${}_{D}\mathsf{CoMod}_{C}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$  is a simplicial model category, homotopy limits over cosimplicial diagrams are computed as in [Hir03, 18.1.8]. We denote the homotopy limit of the cosimplicial diagram  $\Omega^{\bullet}(M, C, C)$  in  ${}_{D}\mathsf{CoMod}_{C}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$  by  $\Omega(M, C, C)$ , and we say it is the *two-sided cobar resolution of* M. By [Pér20, 3.13], we have  $M \simeq \Omega(M, C, C)$  as a (D, C)-bicomodule if M is fibrant as a left D-comodule. Notice that each object in the cosimplicial diagram  $\Omega^{\bullet}(M, C, C)$  is a fibrant right C-comodule by Lemma A.8. Thus  $\Omega(M, C, C)$  is a fibrant (D, C)-bicomodule by [Hir03, 18.5.2] if M is a fibrant left D-comodule.

**Definition 3.9.** Let C, D and E be simply connected dg-coalgebras over  $\Bbbk$ . Let M be a (D, C)-bicomodule and N be a (C, E)-bicomodule. We define the *two-sided cosimplicial cobar construction of* M and N to be the cosimplicial object  $\Omega^{\bullet}(M, C, N)$  in  $_{D}\mathsf{CoMod}_{E}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$  given by  $\Omega^{\bullet}(M, C, C) \square_{C} N$ . We write  $\Omega(M, C, N)$ for the homotopy limit of  $\Omega^{\bullet}(M, C, N)$  in  $_{D}\mathsf{CoMod}_{E}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ , computed as in [Hir03, 18.1.8]. As noted in [Pér20], it is equivalent to compute the homotopy limit in  $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ .

**Proposition 3.10.** Let C, D and E be simply connected dg-coalgebras over  $\Bbbk$ . Let M be a (D, C)-bicomodule and N be a fibrant (C, E)-bicomodule. Then we obtain an equivalence  $\Omega(M, C, C) \square_C N \simeq \Omega(M, C, N)$  as (D, E)-bicomodules.

*Proof.* This argument is similar to that of [Pér20, 4.33] (the cocommutative requirement there was not needed).  $\Box$ 

**Remark 3.11.** By the Dold-Kan correspondence, since C is a simply connected dg-coalgebra, the twosided cosimplicial cobar resolution  $\Omega(M, C, N)$  from Definition 3.8 is quasi-isomorphic to  $\underline{\Omega}(M, C, N)$ , the conormalized cobar resolution of M and N over C, which we now define. We first establish the following notation conventions.

• Given V a graded k-module, we define

$$T(V) = \bigoplus_{n \ge 0} V^{\otimes n}.$$

Elements in the summands are denoted  $v_1 | \cdots | v_n$ , where  $v_i \in V$ .

- Let  $s^{-1}$  denote the desuspension functor on graded k-modules where, for  $V = \bigoplus_{i \in \mathbb{Z}} V_i$ , we define  $(s^{-1}V)_i = V_{i+1}$ . Given a homogeneous element v in V, we write  $s^{-1}v$  for the corresponding element in  $s^{-1}V$ .
- We denote the kernel of the counit  $\varepsilon : C \to \Bbbk$  by <u>C</u>, often referred to as the coideal of C.

The conormalized cobar resolution of C is the chain complex  $\underline{\Omega}C := (T(s^{-1}\underline{C}), d_{\Omega})$  where, if d denotes the differential on C, then

$$d_{\Omega}(s^{-1}c_{1}|\cdots|s^{-1}c_{n}) = \sum_{j=1}^{n} \pm s^{-1}c_{1}|\cdots|s^{-1}(dc_{j})|\cdots|s^{-1}c_{n}$$
$$+ \sum_{j=1}^{n} \pm s^{-1}c_{1}|\cdots|s^{-1}c_{j}|_{(1)}|s^{-1}c_{j}|_{(2)}|\cdots|s^{-1}c_{n}$$

where  $\Delta(c_j) = c_{j(1)} \otimes c_{j(2)}$  denotes the (reduced) comultiplication of <u>C</u> on the element  $c_j$  using the Sweedler notation. To determine the signs above, one needs to apply the Koszul rule. More generally, we define the conormalized cobar resolution of M and N over C to be the chain complex  $\underline{\Omega}(M, C, N) := (M \otimes T(s^{-1}\underline{C}) \otimes N, \delta)$ , where up to Koszul sign, the differential  $\delta$  is defined as

$$\delta(m \otimes s^{-1}c_1|\cdots|s^{-1}c_n \otimes n) = dm \otimes s^{-1}c_1|\cdots|s^{-1}c_n \otimes n$$
  

$$\pm m_{(0)} \otimes s^{-1}m_{(1)}|s^{-1}c_1|\cdots|s^{-1}c_n \otimes n$$
  

$$\pm m \otimes d_{\Omega} \left(s^{-1}c_1|\cdots|s^{-1}c_n\right) \otimes n$$
  

$$\pm m \otimes s^{-1}c_1|\cdots|s^{-1}c_n \otimes dn$$
  

$$\pm m \otimes s^{-1}c_1|\cdots|s^{-1}c_n|s^{-1}n_{(1)} \otimes n_{(0)},$$

where d denotes either the differential on M or N, and  $m \mapsto m_{(0)} \otimes m_{(1)}$  denotes the coaction of  $M \to M \otimes C$ applied to an element m, and  $n \mapsto n_{(1)} \otimes n_{(0)}$  denotes the coaction of  $N \to C \otimes N$  applied to an element n. By the Dold-Kan correspondence, there is an isomorphism of categories between cosimplicial objects and non-positive cochain complexes of bicomodules given by the conormalization functor

$$N^*: \left({}_D\mathsf{CoMod}_E(\mathsf{Ch}_{\Bbbk}^{\geq 0})\right)^{\Delta} \overset{\cong}{\longrightarrow} \mathsf{coCh}^{\leq 0}\left({}_D\mathsf{CoMod}_E(\mathsf{Ch}_{\Bbbk}^{\geq 0})\right).$$

Therefore, we denote  $N^*(\Omega^{\bullet}(M, C, N))$  by  $\underline{\Omega}^{\bullet}(M, C, N)$ . It is a double complex, and one can show that its total complex is precisely  $\underline{\Omega}(M, C, N)$ . A word of warning however: a double complex in general has two possible totalizations, one given using coproducts and one using products (see [Wei94, 1.2.6]). The product-total complex of  $\underline{\Omega}^{\bullet}(M, C, N)$  is quasi-isomorphic to the homotopy limit of  $\Omega^{\bullet}(M, C, N)$ , i.e.,  $\Omega(M, C, N)$  (see [Bun12, 4.23] for instance). On the other hand, the coproduct-total complex of  $\underline{\Omega}^{\bullet}(M, C, N)$  is  $\underline{\Omega}(M, C, N)$ . Of course, if the double complex is bounded, these are equal. For instance, when C is simply-connected, then  $\underline{\Omega}^{-q}(M, C, N)_p = 0$  for  $0 \le p \le 2q - 1$ , and is therefore bounded. Thus when C is simply-connected, we obtain a quasi-isomorphism

$$\Omega(M, C, N) \simeq \underline{\Omega}(M, C, N).$$

But in general, if C is not simply-connected, no such claim can be made.

**Remark 3.12.** Let *C* be a simply-connected coalgebra in  $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ . Let *M* and *N* be left and right *C*-comodules respectively. As noted in [Rav86, A1.2.12], for all  $i \geq 0$  we have an isomorphism

$$\operatorname{CoTor}_{C}^{i}(M,N) \cong H^{i}(\underline{\Omega}^{\bullet}(M,C,N))$$

Therefore, on the homotopy category of comodules, we have a derived cotensor product, which we denote by  $\widehat{\Box}$ . In fact,  $M\widehat{\Box}_C N$  is quasi-isomorphic to the two-sided cobar resolution  $\Omega(M, C, N)$ , see Corollary A.18. The derived cotensor preserves any homotopy coherent coactions, see [Pér20].

Having established the necessary definitions for each of the components, we now define the appropriate bicategory for the derived bicomodule setting.

**Definition 3.13.** Define  $\mathcal{D}\mathsf{CoMod}$  to be the bicategory whose 0-cells are simply connected dg-coalgebras over  $\Bbbk$ , and whose 1-cells, 2-cells, and vertical compositions are given by the homotopy category of  $_C\mathsf{CoMod}_D(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ . The unit  $U_C$  is the fibrant (C, C)-bicomodule C, and horizontal composition is given by the derived cotensor product of bicomodules. Given a fibrant (C, D)-bicomodule M and a fibrant (D, E)-bicomodule N, their horizontal composition  $M \odot N$  is the fibrant (C, E)-bicomodule

$$M\Box_D N \simeq M \Box_D N \simeq \Omega(M, D, N).$$

If P is a fibrant (E, F)-bicomodule, then we define the natural isomorphisms:

$$a: (M \square_D N) \square_E P \xrightarrow{\cong} M \square_D (N \square_E P),$$
$$\ell: C \square_C M \xrightarrow{\cong} M, \quad r: M \square_D D \xrightarrow{\cong} M,$$

as follows. The isomorphism a follows from the natural isomorphism  $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$ , and therefore automatically respects the pentagon identity. The natural isomorphisms  $\ell$  and r are induced by the counits  $C \to \Bbbk$  and  $D \to \Bbbk$  respectively, and the triangle identities follow for the cotensor products since they hold for the tensor product.

Since we shall need it in next section, we provide an explicit definition of a,  $\ell$ , and r from above, where we instead use the cobar construction as a model for the derived cotensor product. The associative equivalence

$$a: \Omega(\Omega(M, D, N), E, P) \xrightarrow{\simeq} \Omega(M, D, \Omega(N, E, P))$$

is induced by an isomorphism of cosimplicial objects  $\Omega^{\bullet}(\Omega^{\bullet}(M, D, N), E, P) \cong \Omega^{\bullet}(M, D, \Omega^{\bullet}(N, E, P))$ , using Corollary A.21 from Appendix A. Indeed, the (i, j)-spot corresponds to the (j, i)-spot up to isomorphism

$$(M \otimes D^{\otimes j} \otimes N) \otimes E^{\otimes i} \otimes P \xrightarrow{\cong} M \otimes D^{\otimes j} \otimes (N \otimes E^{\otimes i} \otimes P)$$

from the usual associative isomorphism. The equivalences  $\ell : \Omega(C, C, M) \xrightarrow{\simeq} M$  and  $r : \Omega(M, D, D) \xrightarrow{\simeq} M$  are defined as in Definition 3.8, and are induced by the counits. For instance, the map  $C^{\otimes n+1} \otimes M \to M$  is induced by repeatedly applying the counit on C.

#### 4. Cohochschild homology as a shadow

Here we show our main results, that coHochschild homology provides a shadow structure on the previously defined bicategories of bicomodules.

**Definition 4.1** ([Pon10, PS13]). A shadow functor for a bicategory  $\mathcal{B}$  consists of functors

$$\langle \langle - \rangle \rangle_C : \mathcal{B}(C, C) \to \mathbb{T}$$

for every  $C \in \mathcal{B}$  and some fixed category **T** equipped with a natural isomorphism for  $M \in \mathcal{B}(C, D)$ ,  $N \in \mathcal{B}(D, C)$ :

$$\theta: \langle \langle M \odot N \rangle \rangle_C \xrightarrow{\cong} \langle \langle N \odot M \rangle \rangle_D$$

For  $P \in \mathcal{B}(C, C)$ , these functors must satisfy the following commutative diagrams

$$\begin{array}{ccc} \langle \langle (M \odot N) \odot P \rangle \rangle_{C} & \xrightarrow{\theta} \langle \langle P \odot (M \odot N) \rangle \rangle_{C} & \xrightarrow{\langle \langle a \rangle \rangle} \langle \langle (P \odot M) \odot N \rangle \rangle_{C} \\ & & & \uparrow^{\theta} \\ \langle \langle M \odot (N \odot P) \rangle \rangle_{C} & \xrightarrow{\theta} \langle \langle (N \odot P) \odot M \rangle \rangle_{D} & \xrightarrow{\langle \langle a \rangle \rangle} \langle \langle N \odot (P \odot M) \rangle \rangle_{D}, \\ & & & \langle \langle P \odot U_{C} \rangle \rangle_{C} & \xrightarrow{\theta} \langle \langle U_{C} \odot P \rangle \rangle_{C} & \longrightarrow \langle \langle P \odot U_{C} \rangle \rangle_{C} \\ & & & \downarrow^{\langle \langle e \rangle \rangle} & \downarrow^{\langle \langle e \rangle \rangle} & & & \langle \langle P \rangle \rangle_{C}. \end{array}$$

In this case, we say  $\mathcal{B}$  is a *shadowed bicategory*, and we write  $(\mathcal{B}, \langle \langle - \rangle \rangle)$  for the bicategory and its shadow.

**Example 4.2.** We can now consider shadows for the bicategories that we introduced in Examples 3.2 and 3.3. For a ring R, the 0<sup>th</sup> Hochschild homology,  $HH_0(R, -)$ , is a shadow on the bicategory with 1- and 2-cells from the category of (R, R)-bimodules to the category of abelian groups, Ab:

$$\langle \langle - \rangle \rangle_R :_R \operatorname{\mathsf{Mod}}_R \to \operatorname{\mathsf{Ab}}$$
  
 $M \mapsto R \otimes_{R \otimes R^{\operatorname{op}}} M \cong \operatorname{HH}_0(R, M).$ 

More generally, a Dennis-Waldhausen Morita argument shows that Hochschild homology, HH(R, -), is a shadow in the derived setting [Wal79]. Further, [BM12] shows that topological Hochschild homology, THH(R, -), is a shadow to the homotopy category of spectra:

$$\langle \langle - \rangle \rangle_R : \operatorname{Ho}(_R \operatorname{\mathsf{Mod}}_R) \to \operatorname{Ho}(\operatorname{Sp})$$
  
 $M \mapsto \operatorname{THH}(R, M).$ 

Since (topological) Hochschild homology provides a bicategorical shadow for the setting of modules, we want to consider the analogue of this construction for the context of comodules. Work of Doi defines coHochschild homology, denoted coHH, as an invariant of coalgebras analogous to Hochschild homology.

**Definition 4.3** ([Doi81]). For a commutative ring R, a coassociative, counital R-coalgebra C, and a (C, C)bicomodule M, build the cochain complex  $\mathcal{H}(M, C)$ :

$$\cdots \longleftarrow M \otimes_R C \otimes_R C \longleftarrow M \otimes_R C \longleftarrow M \longleftarrow 0,$$

as follows. Let  $\mathcal{H}^r(M, C) = M \otimes C^{\otimes r}$  for  $r \ge 0$  with coboundary map  $\delta^r : \mathcal{H}^r(M, C) \to \mathcal{H}^{r+1}(M, C)$  defined by

$$\delta^r = \sum_{i=0}^{r+1} (-1)^i d_i,$$

for  $d_i$  given by

$$d_i = \begin{cases} \rho \otimes \mathrm{id}_C^{\otimes r} & i = 0\\ \mathrm{id}_M \otimes \mathrm{id}_C^{\otimes i-1} \otimes \Delta \otimes \mathrm{id}_C^{\otimes (r-i)} & 1 \le i \le r\\ \tilde{t} \circ (\lambda \otimes \mathrm{id}_C^{\otimes r}) & i = r+1, \end{cases}$$

where  $\rho: M \to M \otimes_R C$  denotes the right coaction,  $\lambda: M \to C \otimes_R M$  denotes the left coaction, and  $\tilde{t}$  is the map that twists the first factor to the last. Then the  $q^{th}$ -coHochschild homology of C with coefficients in M is given by the cohomology of the cochain complex

$$\operatorname{coHH}_q(M, C) := H^q(\mathcal{H}(M, C)).$$

**Remark 4.4.** We can see that  $\mathcal{H}(M, C) = \underline{\Omega}(M, C^e, C)$  as in Remark 3.11, where  $C^e = C \otimes_R C^{op}$ , and in particular we get that  $\operatorname{coHH}_q(M, C) \cong \operatorname{CoTor}_{C^e}^q(M, C)$ , see Definition 1.13 and Remark 3.12. In particular  $\operatorname{coHH}_0(M, C) \cong M \square_{C^e} C.$ 

**Theorem 4.5.** The  $0^{th}$  coHochschild homology, coHH<sub>0</sub>, is a shadow on the bicategory CoMod. That is, it gives a family of functors

$$\operatorname{coHH}_0(-, C) : {}_C\mathsf{CoMod}_C \to \mathsf{Mod}_{\Bbbk}$$
  
 $M \mapsto \operatorname{coHH}_0(M, C)$ 

that satisfy the required shadow properties.

*Proof.* Let C and D be coalgebras over k. Given M a (C, D)-bicomodule and N a (D, C)-bicomodule, we need to show that we have an isomorphism

$$\theta$$
 : coHH<sub>0</sub>( $M\Box_D N, C$ )  $\longrightarrow$  coHH<sub>0</sub>( $N\Box_C M, D$ ).

Using Sweedler notation, we denote the coactions on M as follows:

$$\begin{array}{ll} M \longrightarrow C \otimes M, & M \longrightarrow M \otimes D \\ m \longmapsto m_{(1)}^C \otimes m_{(0)}^C & m \longmapsto m_{(0)}^D \otimes m_{(1)}^D, \end{array}$$

and the coactions on N as follows:

$$\begin{split} N &\longrightarrow D \otimes N, \\ n &\longmapsto n^D_{(1)} \otimes n^D_{(0)} \end{split} \qquad \qquad \begin{split} N &\longrightarrow N \otimes C \\ n &\longmapsto n^C_{(0)} \otimes n^C_{(1)}. \end{split}$$

Recall that  $\operatorname{coHH}_0(M\Box_D N, C)$  is defined as the kernel of

$$\delta^0: M\square_D N \to (M\square_D N) \otimes C.$$

The desired isomorphism will be induced by

$$\tau: M \otimes N \longrightarrow N \otimes M$$
$$m \otimes n \longmapsto n \otimes m.$$

We need to verify that if we restrict  $\tau$  to  $\operatorname{coHH}_0(M \square_D N, C)$ , we indeed corestrict to  $\operatorname{coHH}_0(N \square_C M, D)$ . In other words, we need to verify that if  $m \otimes n \in \operatorname{coHH}_0(M \square_D N, C)$ , then  $n \otimes m \in \operatorname{coHH}_0(N \square_C M, D)$ . Notice that because  $m \otimes n \in \operatorname{coHH}_0(M \square_D N, C)$ , it follows that

- (1)  $m_{(0)}^D \otimes m_{(1)}^D \otimes n = m \otimes n_{(1)}^D \otimes n_{(0)}^D$ , since  $m \otimes n \in M \square_D N$ ; (2)  $m \otimes n_{(0)}^C \otimes n_{(1)}^C = m_{(0)}^C \otimes n \otimes m_{(1)}^C$ , since  $m \otimes n \in \ker(\delta^0)$ .

To see that  $n \otimes m \in \operatorname{coHH}_0(N \square_C M, D)$ , notice that  $n \otimes m \in N \square_C M$  follows from (2) above, while  $n \otimes m \in \ker(\delta^0)$  follows from (1) above. Therefore we have obtained the desired homomorphism,  $\theta$ . An analogous argument defines its inverse, and thus  $\theta$  is an isomorphism.

Next we must show that for a (C, C)-bicomodule P, the following diagram is commutative:

$$\begin{array}{ccc} \operatorname{coHH}_{0}((M\Box_{D}N)\Box_{C}P,C) & \xrightarrow{\theta} \operatorname{coHH}_{0}(P\Box_{C}(M\Box_{D}N),C) & \xrightarrow{\langle\langle a \rangle \rangle} \operatorname{coHH}_{0}((P\Box_{C}M)\Box_{D}N,C) \\ & & & \uparrow^{\theta} \\ \operatorname{coHH}_{0}(M\Box_{D}(N\Box_{C}P),C) & \xrightarrow{\theta} \operatorname{coHH}_{0}((N\Box_{C}P)\Box_{C}M,D) & \xrightarrow{\langle\langle a \rangle \rangle} \operatorname{coHH}_{0}(N\Box_{C}(P\Box_{C}M),D) \end{array}$$

We check its commutativity directly by applying the definition of  $\theta$  given above and a from Definition 3.5:

$$(m \otimes n) \otimes p \longmapsto p \otimes (m \otimes n) \longmapsto (p \otimes m) \otimes n.$$

$$\downarrow \qquad \qquad \uparrow$$

$$m \otimes (n \otimes p) \longmapsto (n \otimes p) \otimes m \longmapsto n \otimes (p \otimes m)$$

Similarly, we need to check the commutativity of the diagram below:



This follows again by applying the definitions of  $\theta$ , r, and  $\ell$  (see Definition 3.5):



This proves that the  $0^{th}$  coHochschild homology is a shadow in this bicategorical setting.

Having established that  $coHH_0$  is a bicategorical shadow on CoMod, we now consider the derived setting. Recall that  $\underline{\Omega}$  of Remark 3.11 is not invariant under quasi-isomorphisms in general. Therefore particular care is required in order to show that coHH is a shadow in the derived setting. To do so, we must instead consider coalgebras in chain complexes that are simply connected. Extending the definitions of [HS21] and [BGH<sup>+</sup>18], the first author introduced in [Kla22, 2.8] the notion of coHochschild homology with coefficients for any model category with a symmetric monoidal structure.

**Definition 4.6.** Let  $(\mathsf{M}, \otimes, \mathbb{I})$  be a symmetric monoidal category with a model structure, and let  $C \in \mathsf{M}$  be a coalgebra with coassociative comultiplication  $\Delta : C \to C \otimes C$  and counit  $\varepsilon : C \to \mathbb{I}$ . Further, let M be a (C, C)-bicomodule with left and right coactions  $\lambda : M \to C \otimes M$  and  $\rho : M \to M \otimes C$  respectively. Define coHH<sup>M</sup> $(M, C)^{\bullet}$  to be the cosimplicial object with r-simplices coHH<sup>M</sup> $(M, C)^r = M \otimes C^{\otimes r}$ , with coface maps

$$d_i = \begin{cases} \rho \otimes \mathrm{id}_C^{\otimes r} & i = 0\\ \mathrm{id}_M \otimes \mathrm{id}_C^{\otimes i-1} \otimes \Delta \otimes \mathrm{id}_C^{\otimes (r-i)} & 1 \le i \le r\\ \tilde{t} \circ (\lambda \otimes \mathrm{id}_C^{\otimes r}) & i = r+1 \end{cases}$$

where  $\tilde{t}$  is the map that twists the first factor to the last, and with codegeneracy maps  $s_i: M \otimes C^{\otimes (r+1)} \to M \otimes C^{\otimes r}$ , for  $0 \leq i \leq r$ ,

$$s_i = \mathrm{id}_M \otimes \mathrm{id}_C^{\otimes i} \otimes \varepsilon \otimes \mathrm{id}_C^{\otimes r-i}.$$

This gives a cosimplicial object of the form

$$\cdots \stackrel{\text{def}}{=} M \otimes C \otimes C \stackrel{\text{def}}{=} M \otimes C \stackrel{\text{def}}{=} M.$$

The coHochschild homology in M of the coalgebra C with coefficients in M is then defined by

$$\operatorname{coHH}^{\mathsf{M}}(M, C) = \operatorname{holim}_{\Delta}(\operatorname{coHH}^{\mathsf{M}}(M, C)^{\bullet})$$

We shall give a higher categorical version of coHochschild homology in Appendix B. Notably, we show in Proposition B.2 that the two definitions agree.

**Remark 4.7.** In the following, we shall always assume  $\mathsf{M} = \mathsf{Ch}_{\Bbbk}^{\geq 0}$ , and thus will omit it from the notation. We show in Proposition B.4 that  $\operatorname{coHH}(M, C) \simeq \Omega(M, C^e, C)$  for C simply connected and M a fibrant (C, C)-bicomodule.

**Theorem 4.8.** CoHochschild homology defines a shadow on the bicategory of derived bicomodules over simply connected coalgebras DCoMod.

*Proof.* Let C and D be simply connected coalgebras in  $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ . Given M a fibrant (C, D)-bicomodule, and N a fibrant (D, C)-bicomodule, we need to show that we have a quasi-isomorphism:

$$\theta$$
 : coHH $(M\widehat{\square}_D N, C) \longrightarrow$  coHH $(N\widehat{\square}_C M, D)$ .

As M and N are fibrant, the derived cotensor  $M\widehat{\square}_D N$  is modeled by  $M\square_D N$  or  $\Omega(M, D, N)$ . We choose the cobar construction as a model: it automatically gives us the desired quasi-isomorphism at the cost of some combinatorics. To provide this equivalence, we apply Corollary A.22 from Appendix A. In particular, we claim that  $\theta$  is induced by an isomorphism of bicosimplicial objects:

$$\theta: \mathrm{coHH}(\Omega^{\bullet}(M, D, N), C)^{\bullet} \xrightarrow{\cong} \mathrm{coHH}(\Omega^{\bullet}(N, C, M), D)^{\bullet}.$$

This is the dual to the Dennis-Waldhausen Morita argument. Indeed, we define  $\theta$  by the usual shuffling isomorphism in the bicosimplicial map below, in which the rows correspond to the two-sided cobar construction, while the columns correspond to the coHochschild complex. For ease of understanding, the diagrams below have been color-coded to illustrate this isomorphism via shuffling.



In particular, the (i, j)-spot in coHH $(\Omega^{\bullet}(M, D, N), C)^{\bullet}$  is isomorphic to the (j, i)-spot in coHH $(\Omega^{\bullet}(N, C, M), D)^{\bullet}$ :

$$\theta: M \otimes D^{\otimes i} \otimes N \otimes C^{\otimes j} \xrightarrow{\cong} N \otimes C^{\otimes j} \otimes M \otimes D^{\otimes i},$$

which incorporates a Koszul sign. Moreover, the map  $\theta$  is compatible with the cofaces and codegeneracies. By [Hir03, 18.5.3], we obtain the equivalence

$$\theta$$
 : coHH( $\Omega(M, D, N), C$ )  $\xrightarrow{\simeq}$  coHH( $\Omega(N, C, M), D$ ),

using the model of homotopy limit as in [Hir03, 18.1.8], providing the desired quasi-isomorphism  $\theta$ .

Given a fibrant (C, C)-bicomodule P, we next must show that the following diagram commutes:

$$\begin{array}{c} \operatorname{coHH}(\Omega(\Omega(M,D,N),C,P),C) \xrightarrow{\theta} \operatorname{coHH}(\Omega(P,C,\Omega(M,D,N)),C) \xrightarrow{\langle\langle a \rangle \rangle} \operatorname{coHH}(\Omega(\Omega(P,C,M),D,N),C) & & & \\ & & & & \\ & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ &$$

This follows a similar argument as above. We need to consider an isomorphism of "tri-cosimplicial" isomorphisms in which we keep track of the swapping of the grading:

$$\begin{array}{c} \left( (M \otimes D^{\otimes k} \otimes N) \otimes C^{\otimes j} \otimes P \right) \otimes C^{\otimes i} \xrightarrow{\theta} \left( P \otimes C^{\otimes i} \otimes (M \otimes D^{\otimes k} \otimes N) \right) \otimes C^{\otimes j} \xrightarrow{\langle \langle a \rangle \rangle} \left( (P \otimes C^{\otimes i} \otimes M) \otimes D^{\otimes k} \otimes N \right) \otimes C^{\otimes j}. \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ \left( (A \otimes D^{\otimes k} \otimes (N \otimes C^{\otimes j} \otimes P) \right) \otimes C^{\otimes i} \xrightarrow{\theta} \left( (N \otimes C^{\otimes j} \otimes P) \otimes C^{\otimes i} \otimes M \right) \otimes D^{\otimes k} \xrightarrow{\langle \langle a \rangle \rangle} \left( N \otimes C^{\otimes j} \otimes (P \otimes C^{\otimes i} \otimes M) \right) \otimes D^{\otimes k} \right) \\ \end{array}$$

We then need to check if the following diagram commutes:



We will prove the left triangle is commutative, but the argument for the right triangle will follow analogously. We apply the definition of  $\theta$ ,  $\ell$ , and r on the bicosimplicial objects:



Here we kept track of the different gradings by  $\bullet$  and \*. Therefore, once we apply the homotopy limit, we obtain the desired result.

## 5. DUALITY

To apply the trace of a shadow in the setting of a bicategory, we must have a notion of a dualizable object as endomorphisms on dualizable objects generalize square matrices.

**Definition 5.1.** Let  $\mathcal{B}$  be a bicategory. A 1-cell  $M \in \mathcal{B}(C, D)$  is *right dualizable* if there is another 1-cell  $M^*$  in  $\mathcal{B}(D, C)$ , with 2-cells  $\eta : U_C \to M \odot M^*$  and  $\varepsilon : M^* \odot M \to U_D$ , called *coevaluation* and *evaluation* respectively, such that the compositions in  $\mathcal{B}(C, D)$  and  $\mathcal{B}(D, C)$  respectively are the identity 2-cells:

$$M \cong U_C \odot M \xrightarrow{\eta \odot \operatorname{id}} (M \odot M^*) \odot M \cong M \odot (M^* \odot M) \xrightarrow{\operatorname{id} \odot \varepsilon} M \odot U_D \cong M,$$
$$M^* \cong M^* \odot U_C \xrightarrow{\operatorname{id} \odot \eta} M^* \odot (M \odot M^*) \cong (M^* \odot M) \odot M^* \xrightarrow{\varepsilon \odot \operatorname{id}} U_D \odot M^* \cong M^*.$$

We call  $M^*$  the right dual of M and say that  $(M, M^*)$  form a dual pair in  $\mathcal{B}$ .

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For instance, in the bicategory of bimodules, an (R, S)-bimodule M is right dualizable if and only if it is finitely generated and projective as a right S-module, and its dual is given by its linear dual  $\text{Hom}_S(M, S)$ , see [PS13, 6.1]. We shall obtain a very similar result for bicomodules with the subtlety that our bicategory is not "closed" (see [Pon10, 4.1.4]), and thus we cannot recognize dualizable objects (as in [Pon10, 4.3.3]) since we are not provided with an internal hom. Our bicategories will almost be "co-closed" thanks to the introduction of a cohom functor (Definition 2.6). **Example 5.2.** Consider the bicategory CoMod of Definition 3.5. A comodule M in  $_{C}CoMod_{\Bbbk}$  is right dualizable if it is quasi-finite (Definition 2.6) and injective as a left C-comodule. The dual of M is then the right C-comodule  $M^* := h_C(M, C)$ , together with the coevaluation  $\eta : C \to M \otimes h_C(M, C)$  and evaluation  $\varepsilon : h_C(M, C) \square_C M \to \Bbbk$  as in Definition 2.9. The desired triangle identities follow from the adjunction of the cohom functor.

**Example 5.3.** In Example 5.2, recall an example of a quasi-finite left *C*-comodule *M* is given by  $M = C \otimes V$ , where *V* is a dualizable k-module (i.e., finitely generated) by Example 2.7. We obtain  $M^* = h_C(C \otimes V, C) \cong V^* \otimes C$ . We can then explicitly define the coevaluation and evaluation. Let us denote the comultiplication and counit of *C* by  $\Delta_C : C \to C \otimes C$  and  $\varepsilon_C : C \to \mathbb{K}$  respectively. Similarly, denote the coevaluation and evaluation of the dualizable k-module *V* by  $\eta_V : \mathbb{K} \to V \otimes V^*$  and  $\varepsilon_V : V^* \otimes V \to \mathbb{K}$  respectively. Then the coevaluation  $\eta : C \to M \square_{\mathbb{K}} M^* \cong (C \otimes V) \otimes (V^* \otimes C)$  is the composite

$$C \xrightarrow{\Delta_C} C \otimes C \cong C \otimes \Bbbk \otimes C \xrightarrow{\mathsf{id} \otimes \eta_V \otimes \mathsf{id}} C \otimes (V \otimes V^*) \otimes C$$

Explicitly, if we write  $(e_1, \ldots, e_n)$  for a basis of V, and  $(e_1^*, \ldots, e_n^*)$  for the dual basis of  $V^*$ , and write  $\Delta_C(c) = c_{(1)} \otimes c_{(2)}$ , we obtain the formula

$$c_{(1)} \otimes \left(\sum_{i=1}^{n} e_i \otimes e_i^*\right) \otimes c_{(2)}.$$

The evaluation  $\varepsilon: M^* \square_C M \cong V^* \otimes C \otimes V \to \Bbbk$  is then the composite

$$V^*\otimes C\otimes V \xrightarrow{\mathrm{id}\otimes \varepsilon_C\otimes \mathrm{id}} V^*\otimes \Bbbk\otimes V\cong V^*\otimes V \xrightarrow{\varepsilon_V} \Bbbk.$$

Explicitly, we obtain the formula

$$\varepsilon(f \otimes c \otimes v) = \varepsilon_C(c)f(v).$$

**Example 5.4.** We now generalize Example 5.2. A comodule M in  $_C$ CoMod<sub>D</sub> is right dualizable whenever it is quasi-finite and injective as a left C-comodule. Define  $M^* = h_C(M, C)$ , which is a (D, C)-bicomodule. Indeed, by [Tak77b, 1.8], for any (C, D)-bicomodule M such that M is quasi-finite as a left C-comodule, and any left C-comodule N, we have that  $h_C(M, N)$  is a left D-comodule with coaction  $h_C(M, N) \to D \otimes h_C(M, N)$  given by the adjoint to the left C-colinear map

$$N \xrightarrow{\eta} M \otimes h_C(M,N) \xrightarrow{\rho \otimes \mathrm{id}} M \otimes D \otimes h_C(M,N)$$

Moreover, by [Tak77b, 1.7], the *C*-bicolinear map  $\eta : C \to M \otimes h_C(M, C)$  factors through the (C, C)bicomodule  $M \Box_D h_C(M, C)$ , and the induced map  $\eta : C \to M \Box_D h_C(M, C)$  remains *C*-bicolinear by [Tak77b, 1.9]. Therefore this defines the desired *C*-bicolinear coevaluation  $\eta : C \to M \Box_D M^*$ . In fact, by [Tak77b, 1.10], the cohom functor  $h_C : {}_C \text{CoMod} \to \text{Mod}_{\Bbbk}$  can be promoted to a functor  $h_C : {}_C \text{CoMod} \to {}_D \text{CoMod}_{\Bbbk}$  whenever the quasi-finite left *C*-comodule *M* is endowed with a right *D*-coaction, and it is the left adjoint of the functor

$${}_D\mathsf{CoMod} \longrightarrow {}_C\mathsf{CoMod}$$
  
 $L \longrightarrow M \Box_D L,$ 

i.e., we obtain an equivalence

$$_{D}\mathsf{CoMod}\Big(h_{C}(M,N),L\Big)\cong {}_{C}\mathsf{CoMod}\Big(N,M\Box_{D}L\Big)$$

for any left C-comodule N and left D-comodule L. In particular, if we choose M = N and L = D, the adjoint of the identity map on M provides a left D-colinear map  $h_C(M, M) \to D$ , which is in fact D-bicolinear by [Tak77b, 1.11]. Moreover, by [Tak77b, 1.15], the map

$$h_C(M,M) \cong h_C(M,C\square_C M) \xrightarrow{\partial} h_C(M,C)\square_C M$$

is D-bicolinear. When M is injective as a left C-comodule, the map  $\partial$  is an isomorphism. Therefore, the desired evaluation  $\varepsilon: M^* \square_C M \to D$  is given by composing the previous D-bicolinear maps

$$h_C(M,C) \square_C M \xrightarrow{\partial^{-1}} h_C(M,C \square_C M) \cong h_C(M,M) \longrightarrow D.$$

Just as in Example 5.2, the triangle identities follow from the adjunction of the cohom functor.

**Example 5.5.** A particular case of Example 5.4 is when  $C = \Bbbk$ . A bicomodule M in  $\Bbbk CoMod_D$  is right dualizable when M is dualizable as a  $\Bbbk$ -module (i.e., finitely generated), and its right dual is  $M^*$ , the usual linear dual just as above in Definition 2.25.

We can further generalize the examples above to the differential graded case. Arguments for quasi-finite comodules are entirely categorical (see [Al-02]). However, the notion has not been documented before in this context, and therefore we introduce them here.

**Definition 5.6.** Let C be a simply connected dg-coalgebra over  $\Bbbk$ . A left dg-C-comodule M is quasi-finite if the functor

$$\mathsf{Ch}^{\geq 0}_{\Bbbk} \longrightarrow {}_{C}\mathsf{CoMod}$$
$$V \longmapsto M \otimes V$$

admits a left adjoint, denoted  $h_C(M, -) : {}_C\mathsf{CoMod} \to \mathsf{Ch}^{\geq 0}_{\Bbbk}$ , called the *cohom functor*. In other words, we obtain an equivalence

$$\mathsf{Ch}^{\geq 0}_{\Bbbk}\Big(h_C(M,N),V\Big) \cong {}_C\mathsf{CoMod}\Big(N,M\otimes V\Big),$$

for any left dg-C-comodule N and  $\Bbbk$ -chain complex V.

**Example 5.7.** If V is a perfect k-chain complex, then just as in Example 2.7, we have that  $C \otimes V$  is quasi-finite, and  $h_C(C \otimes V, N) \cong V^* \otimes N$ . If C = k, then a left dg C-comodule is just a chain complex, and quasi-finite comodules are precisely the perfect chain complexes.

If M is also endowed with a right D-coaction, then  $h_C(M, N)$  is a right D-comodule and we obtain the adjunction

$$_{C}\mathsf{CoMod} \xrightarrow[]{h_{C}(M,-)}{\underset{M\square_{D}-}{\perp}} _{D}\mathsf{CoMod}.$$

Therefore, just as in the discrete case, the right dual of a (C, D)-bicomodule M should be the (D, C)bicomodule  $h_C(M, C)$ . Notice, even if M is a fibrant (C, D)-bicomodule, there is no reason to expect  $h_C(M, C)$  is fibrant as a (D, C)-bicomodule. However, this is not needed. Indeed, by Proposition A.17, if M is a fibrant (C, D)-bicomodule, then

$$\begin{split} M \Box_D h_C(M,C) &\simeq M \Box_D \Omega(D,D,h_C(M,C)) \\ &\simeq \Omega(M,D,h_C(M,C)) \\ &\simeq M \widehat{\Box}_D h_C(M,C). \end{split}$$

Similarly, we obtain  $h_C(M, C) \square_C M \simeq h_C(M, C) \widehat{\square}_C M$ .

Just as in the discrete case, the C-bicolinear map  $\eta : C \to M \square_D h_C(M, C)$  from the unit of the cohom adjunction induces the desired coevaluation on the dual pair  $(M, h_C(M, C))$ .

We now describe how to obtain the evaluation. Recall that a k-chain complex V is perfect if and only if the natural map  $V \otimes \mathsf{Hom}_{\Bbbk}(V, \Bbbk) \to \mathsf{Hom}_{\Bbbk}(V, V)$  is an isomorphism. Similarly, as in Example 5.2, for any quasi-finite left C-comodule M, we have a natural D-bicolinear map  $\partial : h_C(M, M) \to h_C(M, C) \square_C M$ . We saw in the discrete case that if M is injective and quasi-finite, then  $\partial$  is an isomorphism.

**Definition 5.8.** A fibrant left *C*-comodule *M* is said to be *coperfect* if it is quasi-finite and the induced map  $\partial : h_C(M, M) \to h_C(M, C) \square_C M$  is an isomorphism.

Given a coperfect (C, D)-bicomodule M, define the evaluation as

$$\varepsilon: h_C(M, C) \square_C M \xrightarrow{\partial^{-1}} h_C(M, M) \longrightarrow D.$$

Combining our arguments above, we obtain the desired triangle identities by adjunction of the cohom functor, and thus the following result.

**Proposition 5.9.** Let C and D be simply connected dg-coalgebras. A (fibrant) (C, D)-bicomodule M is right dualizable if it is coperfect as a left C-comodule. Its right dual is given by  $h_C(M, C)$ .

**Example 5.10.** Suppose  $D = \mathbb{k}$ , and  $M = C \otimes V$ , where V is a perfect chain complex. Then M is quasi-finite as a left C-comodule and  $M^* = h_C(M, C) \cong V^* \otimes C$ . Moreover, M is coperfect because

$$V^* \otimes M \cong h_C(C \otimes V, M) \to h_C(C \otimes V, C) \Box_C M \cong V^* \otimes M$$

is an isomorphism.

**Example 5.11.** If  $C = \mathbb{k}$ , then if M is a right D-comodule such that M is a perfect chain complex, then M is dualizable and  $M^* = \text{Hom}_{\mathbb{k}}(M, \mathbb{k})$ .

## 6. Traces

Every shadowed bicategory defines a notion of traces on its dualizable objects. This extends the notion of traces on symmetric monoidal categories, as reviewed in [PS14]. We first recall the general definition of a trace.

**Definition 6.1.** [PS13] Let  $(\mathcal{B}, \langle \langle - \rangle \rangle)$  be a shadowed bicategory. Let  $M \in \mathcal{B}(C, D)$  be a right dualizable 1-cell. The *trace* of a 2-cell  $f : M \to M$ , denoted  $\operatorname{tr}^{\mathcal{B}}(f)$ , is the composite

$$\langle \langle U_C \rangle \rangle_C \xrightarrow{\langle \langle \eta \rangle \rangle} \langle \langle M \odot M^\star \rangle \rangle_C \xrightarrow{\langle \langle f \odot \mathsf{id} \rangle \rangle} \langle \langle M \odot M^\star \rangle \rangle_C \xrightarrow{\theta} \langle \langle M^\star \odot M \rangle \rangle_D \xrightarrow{\langle \langle \varepsilon \rangle \rangle} \langle \langle U_D \rangle \rangle_D$$

More generally, given 1-cells  $P \in \mathcal{B}(D, D)$  and  $Q \in \mathcal{B}(C, C)$ , the trace of a 2-cell  $f : Q \odot M \to M \odot P$  is the composite

$$\langle\langle Q \rangle\rangle_C \xrightarrow{\langle\langle \mathsf{id} \odot \eta \rangle\rangle} \langle\langle Q \odot M \odot M^\star \rangle\rangle_C \xrightarrow{\langle\langle f \odot \mathsf{id} \rangle\rangle} \langle\langle M \odot P \odot M^\star \rangle\rangle_C \xrightarrow{\theta} \langle\langle M^\star \odot M \rangle\rangle_D \xrightarrow{\langle\langle \varepsilon \odot \mathsf{id} \rangle\rangle} \langle\langle P \rangle\rangle_D.$$

The Euler characteristic of M, denoted  $\chi(M)$ , is the trace of its identity 2-cell tr<sup>B</sup>(id<sub>M</sub>).

Bicategorical traces enjoy many useful properties that are recorded in [PS13, Section 7]. Notably, we obtain the following cyclicity property.

**Proposition 6.2** ([PS13, 7.3]). If M and N are right dualizable 1-cells in a shadowed bicategory  $\mathcal{B}$  and if  $f: M \to N$  and  $g: N \to M$  are 2-cells in  $\mathcal{B}$ , then  $\operatorname{tr}^{\mathcal{B}}(f \circ g) = \operatorname{tr}^{\mathcal{B}}(g \circ f)$ .

**Example 6.3** ([Pon10, 4.2.2]). Let R be a ring and let M be a finitely generated right R-module. Let f be an R-linear endomorphism on M. The Hattori-Stalling trace can be regarded as a bicategorical trace  $\operatorname{tr}(f) : \mathbb{Z} \to \operatorname{HH}_0(R)$ , i.e.,  $\operatorname{tr}(f) \in \operatorname{HH}_0(R)$ .

**Example 6.4.** Let C be a k-coalgebra. Given a dualizable left C-comodule M, the trace of a C-colinear endomorphism  $f: M \to M$  is then the Hattori-Stallings cotrace  $\operatorname{coHH}_0(C) \to \mathbb{k}$ , as in Definition 2.11.

**Remark 6.5.** In upcoming work of Justin Barhite, Hochschild cohomology is shown to be a so-called coshadow structure on the bicategory of bimodules over a k-algebra A. He obtains what is called a cotrace  $\operatorname{HH}^n(A) \to \operatorname{End}_A(M)$  for any A-linear endomorphism on a finitely generated projective right A-module M. If C is a k-coalgebra that is finitely generated as k-module, then the bicategory of bimodules over  $C^*$  is intimately related to the bicategory of bicomodules over C since  $M \square_C N \cong_{C^*} \operatorname{Hom}_{C^*}(C^*, M \otimes N)$  by [BW03, 10.10]. From our perspective however, our cotrace comes from a shadow and not a coshadow, and we do not assume that our bicategory is closed. We see that we recover his cotrace  $\operatorname{HH}^n(A) \to \operatorname{End}_A(M)$  when n = 0 and  $A = C^*$ . Nonetheless, it does not seem that the language of coshadows is the appropriate vocabulary for the Hattori-Stallings cotrace, nor the C-trace.

**Example 6.6.** Let C be a k-coalgebra. Given a right C-comodule M that is finitely generated as a k-module, recall that  $M \square_C M^* \cong \operatorname{Hom}_C(M, M)$ , and  $\operatorname{coHH}_0(M \square_C M^*, \Bbbk) \cong M \square_C M^*$ . The trace of a C-colinear

endomorphism  $f: M \to M$  is a k-linear map  $k \to \operatorname{coHH}_0(C)$  given by

$$\begin{array}{ccc} & & \stackrel{\eta}{\longrightarrow} \operatorname{Hom}_{C}(M, M) \\ & & & \downarrow^{f_{*}} \\ & & & \operatorname{Hom}_{C}(M, M) \\ & & \cong \downarrow^{\theta} \\ & & & \operatorname{coHH}_{0}(M^{*} \otimes M, C) \xrightarrow{\operatorname{coHH}_{0}(\varepsilon)} \operatorname{coHH}_{0}(C). \end{array}$$

This is the Hattori-Stallings C-trace for f as in Definition 2.25.

**Example 6.7.** We can generalize the Hattori-Stallings cotrace for chain complexes. Let C be a simply connected dg-coalgebra over  $\Bbbk$ , let M be a coperfect left C-comodule, and let f be an C-colinear endomorphism on M. Then we obtain the cotrace

$$\begin{array}{ccc} \operatorname{coHH}(C) & \xrightarrow{\operatorname{coHH}(\eta)} & \operatorname{coHH}(M \otimes M^*, C) \\ & & & \downarrow_{\operatorname{coHH}(f \otimes \operatorname{id})} \\ & & \operatorname{coHH}(M \otimes M^*, C) \\ & & \cong \downarrow \theta \\ & & & M^* \widehat{\square}_C M \xrightarrow{\varepsilon} & \Bbbk. \end{array}$$

Since the homomorphism is in  $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ , and  $\Bbbk$  is concentrated in degree zero, then the above trace is entirely determined by  $\operatorname{coHH}_0(C)$ . Unfortunately, as C is simply connected,  $\operatorname{coHH}_0(C) = \Bbbk$ , and this cotrace is simply the alternating sum of the usual trace of f on each degree. Similarly, we can also generalize the C-trace for chain complexes. Letting V denote a perfect chain complex with a right C-comodule structure, given a C-colinear endomorphism  $f: V \to V$ , its C-trace will define a chain homomorphism  $\Bbbk \to \operatorname{coHH}(C)$ . It is non-trivial only in degree zero, but as C is simply connected, we get that the C-trace is the alternating sum of the trace of f on each degree.

**Example 6.8.** Let X be a simply connected CW-complex, and let Y be a finite CW-complex. Let  $C_*(-; \Bbbk) = C_*(-)$  denote the singular chain complex over  $\Bbbk$  associated to a space. Recall that  $C_*(X)$  is a dg-coalgebra over  $\Bbbk$  using Alexander-Whitney formula and the diagonal  $X \to X \times X$ . Then  $C_*(X \times Y) \cong C_*(X) \otimes C_*(Y)$  is a coperfect  $C_*(X)$ -comodule. Let  $f: Y \to Y$  be an endomorphism. This defines an endomorphism on  $X \times Y$  that is the identity on X, and thus an endomorphism  $\widehat{f}$  on the comodule  $C_*(X \times Y)$ . Therefore the cotrace of  $\widehat{f}$  is a map coHH( $C_*(X)$ )  $\to \Bbbk$  in which its image is the usual Lefschetz number of f.

**Example 6.9.** Let X be a simply connected CW-complex, and let Y be a finite CW-complex with a continuous map  $\rho: Y \to X$ . Then  $C_*(Y)$  is a comodule over  $C_*(X)$ . Let f be an endomorphism over Y. Then the  $C_*(X)$ -trace induced on coHochschild homology  $\Bbbk \to \operatorname{coHH}(C_*(X))$  is the usual alternating sum of the traces of  $C_*(Y) \xrightarrow{f} C_*(Y) \xrightarrow{\rho} C_*(X)$ .

# 7. Morita-Takeuchi invariance

In bicategories, Morita equivalence is the natural notion of an equivalence in a bicategory. It extends the usual notion of equivalence of categories and Morita equivalence between rings.

**Definition 7.1** ([CP19, 4.1, 4.2]). Let  $\mathcal{B}$  be a bicategory, let  $M \in \mathcal{B}(C, D)$  be a right dualizable 1-cell, and denote its right dual by  $M^*$ . We say the dual pair  $(M, M^*)$  is a *Morita equivalence* in  $\mathcal{B}$  if the coevaluation  $\eta : U_C \to M \odot M^*$  and evaluation  $\varepsilon : M^* \odot M \to D$  maps are isomorphisms in  $\mathcal{B}(C, C)$  and  $\mathcal{B}(D, D)$  respectively. If such a pair exists, we say C and D are *Morita equivalent*.

**Proposition 7.2** ([CP19, 4.5]). Let  $(\mathcal{B}, \langle \langle - \rangle \rangle)$  be a shadowed bicategory. Let  $M \in \mathcal{B}(C, D)$  be a right dualizable 1-cell and denote its right dual by  $M^*$ . If  $(M, M^*)$  is a Morita equivalence, then the Euler

characteristic  $\chi(M)$  is an isomorphism, with inverse  $\chi(M^*)$ . In particular, if C and D are Morita equivalent, then

$$\langle \langle U_C \rangle \rangle_C \cong \langle \langle U_D \rangle \rangle_D.$$

**Example 7.3.** Consider the bicategory CoMod of bicomodules over k as in Definition 3.5. Then in [Tak77b, 2.3], a Morita equivalence is referred to as a set of equivalence data. It is then shown than a Morita equivalence in this bicategory recovers the notion of Morita-Takeuchi equivalence in k-modules (Definition 1.11). Therefore, if C and D are Morita-Takeuchi equivalent, then  $\operatorname{coHH}_0(C) \cong \operatorname{coHH}_0(D)$  by Proposition 7.2. This recovers the results of [FS98] at level zero. By [Tak77b, 3.5], if M is a left C-comodule that is a quasi-finite injective cogenerator, and  $D := e_C(M)$ , the coalgebra of coendomorphisms, then C and D are Morita-Takeuchi equivalence in the bicategory CoMod.

**Example 7.4.** Consider the bicategory of derived bicomodules  $\mathcal{D}$ CoMod as in Definition 3.13. We say two simply connected coalgebras C and D in  $Ch_{\mathbb{k}}^{\geq 0}$  are *homotopically Morita-Takeuchi equivalent* if they are Morita equivalent in the bicategory  $\mathcal{D}$ CoMod. In this situation

$$\operatorname{coHH}(C) \cong \operatorname{coHH}(D).$$

This extends the results of [HS21], where we suspect that the simply-connected condition was forgotten to be mentioned.

**Proposition 7.5.** Suppose  $(M, M^*)$  form a dual pair in  $\mathcal{D}CoMod$ . Then we obtain a Quillen adjunction

$$M^{\star}\square_{C}-: {}_{C}\mathsf{CoMod}(\mathsf{Ch}_{\Bbbk}^{\geq 0}) \xrightarrow{\perp} {}_{D}\mathsf{CoMod}(\mathsf{Ch}_{\Bbbk}^{\geq 0}): M\square_{D}-.$$

Moreover, it is a Quillen equivalence if and only if  $(M, M^*)$  is a homotopical Morita-Takeuchi equivalence.

Proof. Since M and  $M^*$  are dual to each other, the maps  $\eta : C \to M \square_D M^*$  and  $\varepsilon : M^* \square_C M \to D$ induce the adjunction. For instance, given a left C-comodule P, a left D-comodule Q, and a D-collinear map  $\alpha : M^* \square_C P \to Q$ , we obtain a C-collinear map  $P \to M \square_D Q$  via the composite

$$P \cong C \square_C P \xrightarrow{\eta \square \mathsf{id}} (M \square_D M^\star) \square_C P \cong M \square_D (M^\star \square_C P) \xrightarrow{\mathsf{id} \square \alpha} M \square_D Q.$$

The axioms on  $\eta$  and  $\varepsilon$  guarantee that the procedure gives a correspondence:

$$_{D}\mathsf{CoMod}(\mathsf{Ch}^{\geq 0}_{\Bbbk})\Big(M^{\star}\Box_{C}P,Q\Big)\cong {_{C}\mathsf{CoMod}}(\mathsf{Ch}^{\geq 0}_{\Bbbk})\Big(P,M\Box_{D}Q\Big).$$

Since the cotensor product is left exact, then the functor  $M^*\square_D$  preserves monomorphisms, i.e., cofibrations. Since  $M^*$  is a fibrant right *C*-comodule, then  $M^*\square_C$  preserves quasi-isomorphisms, i.e., weak equivalences (by Proposition A.17). Therefore we obtain that it is a Quillen adjunction.

Suppose  $(M, M^*)$  is a homotopical Morita-Takeuchi equivalence and  $P \to P'$  is a C-colinear map such that the induced map

$$M^* \square_C P \xrightarrow{\simeq} M^* \square_C P'$$

is a quasi-isomorphism. Then if we apply  $M \square_D -$ , since M is a fibrant D-comodule, we obtain the quasiisomorphism by Proposition A.17:

$$\underbrace{(M \Box_D M^*) \Box_C P}_{\cong P} \cong M \Box_D (M^* \Box_C P) \xrightarrow{\simeq} M \Box_D (M^* \Box_C P') \cong \underbrace{(M \Box_D M^*) \Box_C P'}_{\cong P'}.$$

Therefore  $P \to P'$  is a quasi-isomorphism, and thus  $M^* \square_C -$  reflects quasi-isomorphism. Moreover, for any fibrant left *D*-comodule *Q*, we have a quasi-isomorphism

$$Q \cong D \square_D Q \simeq M^* \square_C M \square_D Q \longrightarrow Q.$$

Thus by [Hov99, 1.3.16], we obtain that the adjunction is a Quillen equivalence.

Conversely, if we supposed the adjunction to be a Quillen equivalence, then if we apply the adjoints on C and D, we recover that the coevaluation and evaluation  $C \to M \square_D M^*$  and  $M^* \square_C M \to D$  are quasi-isomorphisms by again applying [Hov99, 1.3.16].

**Example 7.6.** Given a map of simply connected coalgebras  $f: C \to D$ , we recover the expected result that if f is a quasi-isomorphism, then C and D are homotopically Morita-Takeuchi equivalent. In more details, recall that (C, C) form a dual pair of bicomodules over C, and  $h_C(C, C) \cong C$ . Moreover, C can be regarded as a left D-comodule via f:

$$C \longrightarrow C \otimes C \xrightarrow{f \otimes \mathsf{id}} D \otimes C.$$

In fact, any left C-comodule P can be regarded as a left D-comodule this way, we shall denote it by  $f^*(C)$ . Notice that  $f^*(C) \cong C \square_C P$  as a left D-comodule. We obtain a Quillen adjunction:

$$f^*: {}_{C}\mathsf{CoMod}(\mathsf{Ch}_{\Bbbk}^{\geq 0}) \xrightarrow{\perp} {}_{D}\mathsf{CoMod}(\mathsf{Ch}_{\Bbbk}^{\geq 0}): C \Box_{D} -,$$

which is a Quillen equivalence if and only if f is a quasi-isomorphism. This recovers the usual change of coalgebras, see [Pér20, 4.29].

# APPENDIX A. HOMOTOPY THEORY OF CONNECTIVE BICOMODULES

The goal of the appendix is to show that the homotopy theory of bicomodules is endowed with a derived cotensor product which provides a homotopy coherent monoidal structure.

Let  $\mathcal{C}^{\otimes}$  be a symmetric monoidal  $\infty$ -category, as in [Lur17, 2.0.0.7]. Subsequently, we only refer to its underlying  $\infty$ -category,  $\mathcal{C}$ , as in [Lur17, 2.1.2.20]. By an  $\mathbb{A}_{\infty}$ -algebra in  $\mathcal{C}$ , we mean an associative algebra in the sense of [Lur17, 4.1.1.6]. Given  $\mathbb{A}_{\infty}$ -algebras A and B in  $\mathcal{C}$ , denote the  $\infty$ -category of (A, B)-bimodule objects in  $\mathcal{C}$  by  $_A \operatorname{\mathsf{Mod}}_B(\mathcal{C})$ , as in [Lur17, Section 4.3]. From [Pér22a, 2.1], define an  $\mathbb{A}_{\infty}$ -coalgebra in  $\mathcal{C}$  to be an  $\mathbb{A}_{\infty}$ -algebra in the opposite category  $\mathcal{C}$ . Similarly as in [Lur17, 4.1.1.7], given an  $\mathbb{A}_{\infty}$ -coalgebra C in  $\mathcal{C}$ , we can define  $C^{\operatorname{op}}$ , the opposite  $\mathbb{A}_{\infty}$ -coalgebra of C.

**Definition A.1.** Let  $\mathcal{C}$  be a symmetric monoidal  $\infty$ -category. Let C and D be  $\mathbb{A}_{\infty}$ -coalgebras in  $\mathcal{C}$ . A (C, D)-bicomodule object in  $\mathcal{C}$  is a (C, D)-bimodule object in  $\mathcal{C}^{op}$ . The  $\infty$ -category of (C, D)-bicomodules in  $\mathcal{C}$  is defined as

$${}_{C}\mathsf{CoMod}_{D}(\mathcal{C}) := ({}_{C}\mathsf{Mod}_{D}(\mathcal{C}^{\mathsf{op}}))^{\mathsf{op}}$$

We define the  $\infty$ -categories of left C-comodules  $_{C}\mathsf{CoMod}(\mathcal{C})$  and right D-comodules  $\mathsf{CoMod}_{D}(\mathcal{C})$  similarly.

**Remark A.2.** Just as in Remark 1.9, given  $\mathbb{A}_{\infty}$ -coalgebras C and D in a symmetric monoidal  $\infty$ -category C, we obtain an equivalence of  $\infty$ -categories

$${}_{C}\mathsf{CoMod}_{D}(\mathcal{C})\simeq\mathsf{CoMod}_{C^{\mathsf{op}}\otimes D}(\mathcal{C}).$$

This follows from [Lur17, 4.6.3.11] applied to the opposite category. Notice that the statement requires the monoidal product of C to commute with totalizations. However, as noted above [Lur17, 4.6.3.3], this requirement is not essential and remains true for any symmetric monoidal  $\infty$ -category C.

**Remark A.3.** Let C be a symmetric monoidal category. Let  $C^{\otimes}$  be its operator category as in [Lur17, 2.0.0.1]. Then its nerve  $\mathcal{N}(C^{\otimes})$  is a symmetric monoidal  $\infty$ -category whose underlying  $\infty$ -category is  $\mathcal{N}(C)$ , see [Lur17, 2.1.2.21]. Let C be a coalgebra in C. It can be regarded as an  $\mathbb{A}_{\infty}$ -coalgebra in  $\mathcal{N}(C)$ , see [Pér22a, 2.3]. If D is also a coalgebra in C, then we obtain an equivalence of  $\infty$ -categories

$$\mathcal{N}(_{C}\mathsf{CoMod}_{D}(\mathsf{C})) \simeq {_{C}\mathsf{CoMod}_{D}(\mathcal{N}(\mathsf{C}))}.$$

Let M be a symmetric monoidal model category as in [Hov99, 4.2.6], with a class of weak equivalence denoted W. We assume every object is cofibrant. Its Dwyer-Kan localization is the  $\infty$ -category denoted  $\mathcal{N}(\mathsf{M})[\mathsf{W}^{-1}]$  following [Lur17, 1.3.4.15] whose homotopy category is the homotopy category of M. It is endowed with a symmetric monoidal structure via the derived tensor product, see [Lur17, 4.1.7.6]. Suppose now that the model structure on M is combinatorial. Given algebras A and B in M, the category of bimodules  ${}_{A}\mathsf{Mod}_{B}(\mathsf{M})$  is endowed with a model structure whose class of weak equivalences, denoted W', are the morphisms of bimodules which are weak equivalences regarded as in M. By [Lur17, 4.3.3.17], we obtain an equivalence of  $\infty$ -categories

$$\mathcal{N}({}_{A}\mathsf{Mod}_{B}(\mathsf{M}))[\mathsf{W}'^{-1}] \simeq {}_{A}\mathsf{Mod}_{B}(\mathcal{N}(\mathsf{M})[\mathsf{W}^{-1}]).$$

The same statement for bicomodules is challenging if we insist on considering a combinatorial monoidal model category (and not "co-combinatorial"). Following [Pér20], we show in Theorem A.4 when the equivalence

above does hold for bicomodules. We denote the Dwyer-Kan localization of  $\mathsf{Ch}_{\Bbbk}^{\geq 0}$  by  $\mathcal{D}^{\geq 0}(\Bbbk)$ ; it is equivalent to the symmetric monoidal  $\infty$ -category of connective Hk-modules in spectra.

**Theorem A.4.** Let C and D be simply connected dg-coalgebras over  $\Bbbk$ . Then the natural functor

$$\mathcal{N}\left({}_{C}\mathsf{CoMod}_{D}(\mathsf{Ch}^{\geq 0}_{\Bbbk})
ight)\left[\mathsf{W}^{-1}
ight]\longrightarrow {}_{C}\mathsf{CoMod}_{D}(\mathcal{D}^{\geq 0}(\Bbbk))$$

is an equivalence of  $\infty$ -categories.

*Proof.* The result was proved in [Pér20, 1.1] for right comodules. We can deduce the result for bicomodules using Remarks 1.9 and A.2. In more details, if C and D are simply connected, then so is  $C \otimes D$ :

$$(C\otimes D)_0=C_0\otimes D_0\cong \Bbbk\otimes \Bbbk\cong \Bbbk, \quad (C\otimes D)_1=(C_1\otimes D_0)\oplus (C_0\otimes D_1)=0.$$

Therefore we can apply [Pér20, 1.1] to the model category of right  $(C^{op} \otimes D)$ -comodules. In particular, we obtain the following diagram of  $\infty$ -categories:

Here, we let W' denote the class of right  $(C^{op} \otimes D)$ -comodule morphisms that are quasi-isomorphisms in  $Ch_{\Bbbk}^{\geq 0}$ . By [Pér20, 3.3], the right vertical map on the diagram above is an equivalence of  $\infty$ -categories. By Remark 1.9, the top horizontal map is an equivalence. By Remark A.2, the bottom horizontal map is an equivalence. Thus the left vertical map is an equivalence of  $\infty$ -categories.

We now describe fibrant objects in  ${}_{C}\mathsf{CoMod}_{D}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$  using Postnikov towers. We say a tower  $\{M(n)\}$  in  ${}_{C}\mathsf{CoMod}_{D}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$  stabilizes in each degree if for all  $n \geq 0$ , and all  $0 \leq i \leq n$ , we obtain isomorphisms of  $\Bbbk$ -modules

$$M(n+1)_i \cong M(n+2)_i \cong M(n+3)_i \cong \cdots$$

Although in general non-finite limits in  ${}_{C}\mathsf{CoMod}_{D}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$  do not correspond to the underlying limit in  $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ , limits of towers that stabilizes in each degree in  ${}_{C}\mathsf{CoMod}_{D}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$  do correspond to their underlying limits, see [Pér21, 4.20]. Furthermore, usually the functor  $M\square_{C}-: {}_{C}\mathsf{CoMod}(\mathsf{Ch}_{\Bbbk}^{\geq 0}) \to \mathsf{Ch}_{\Bbbk}^{\geq 0}$  does not preserve non-finite limits, but it does for towers that stabilize in each degree, see [Pér21, 4.28].

**Proposition A.5** ([Pér20, 2.18]). Let C and D be simply connected dg-coalgebras over k. Let M be a (C, D)-bicomodule in  $Ch_{\Bbbk}^{\geq 0}$ . There exists a tower  $\{M(n)\}$  in  $_{C}CoMod_{D}(Ch_{\Bbbk}^{\geq 0})$  that stabilizes in each degree defined as follows:

- M(0) = 0;
- $M(1) = C \otimes M \otimes D;$
- if the (C, D)-bicomodule M(n) is defined together with a cofibration  $M \hookrightarrow M(n)$  that induces an isomorphism  $H_i(M) \cong H_i(M(n))$  for  $0 \le i \le n$ , then M(n+1) and the cofibration  $M \hookrightarrow M(n+1)$  are defined by the pullback (both in  $_C \text{CoMod}_D(\mathsf{Ch}_{\Bbbk}^{\ge 0})$  and in  $\mathsf{Ch}_{\Bbbk}^{\ge 0}$ ):



where the right vertical map is the (C, D)-cofree map induced by an epimorphism  $P \to Q$  in  $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ . Moreover  $\operatorname{holim}_n M(n) \simeq \lim_n M(n) \cong M$ , and the (homotopy) limits are determined in  $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ . We now show that fibrant objects in the category of bicomodules behave well with respect to the cotensor product.

Consider the general case of a symmetric monoidal category,  $(\mathsf{C}, \otimes, \mathbb{I})$ , that is abelian, and consider flat coalgebras C and D in  $\mathsf{C}$ . Then the category of bicomodules  ${}_{C}\mathsf{CoMod}_{D}(\mathsf{C})$  remains abelian, and the forgetful functor  $U : {}_{C}\mathsf{CoMod}_{D}(\mathsf{C}) \to \mathsf{C}$  preserves and reflects exact sequences, kernels, cokernels, monomorphisms, and epimorphisms.

**Lemma A.6** ([Pér20, 4.22]). Let C and D be dg-coalgebras over k. Let  $f : M \to N$  be an epimorphism in  $_{C}CoMod_{D}(Ch_{\Bbbk}^{\geq 0})$ , and let F be its kernel. Then f is a fibration in  $_{C}CoMod_{D}(Ch_{\Bbbk}^{\geq 0})$  if and only if F is fibrant in  $_{C}CoMod_{D}(Ch_{\Bbbk}^{\geq 0})$ .

**Lemma A.7.** Let C and D be simply connected dg-coalgebras over  $\Bbbk$ . If M is a fibrant (C, D)-bicomodule, then M is also a fibrant left C-comodule and a fibrant right D-comodule.

*Proof.* Since M is a fibrant (C, D)-bicomodule, it is a retract of the limit  $\tilde{X}$  of its Postnikov tower  $\{M(n)\}$ . Since for all chain complexes V, the bicomodule  $C \otimes V \otimes D$  is fibrant both as a left C-comodule and right D-comodule, then by Lemma A.6, we can conclude the desired result.

**Lemma A.8.** Let V be in  $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ , let C and D be simply connected dg-coalgebras over  $\Bbbk$ , and let N be a fibrant right D-comodule. Then  $(C \otimes V) \otimes N$  is a fibrant (C, D)-bicomodule.

Proof. We consider  $\{N(m)\}$ , the Postnikov tower in  $\mathsf{CoMod}_D(\mathsf{Ch}_{\Bbbk}^{\geq 0})$  of N. As N is a retract of the limit  $\widetilde{N} = \lim_{m \to \infty} N(m)$ , then it is enough to show that  $C \otimes V \otimes \widetilde{N}$  is a fibrant (C, D)-bicomodule. Therefore we need to prove that  $\{(C \otimes V) \otimes N(m)\}$  is a fibrant tower of (C, D)-bicomodules. For m = 0, we get  $(C \otimes V) \otimes N(0) = 0$ , which is fibrant. For m = 1, we get  $(C \otimes V) \otimes N(1) = (C \otimes V) \otimes N \otimes D$ , which is a cofree (C, D)-bicomodule and is thus a fibrant (C, D)-bicomodule. Suppose  $m \geq 1$ , then we obtain the pullback in  ${}_{C}\mathsf{CoMod}_{D}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ :

$$\begin{array}{ccc} (C \otimes V) \otimes N(m+1) \longrightarrow (C \otimes V) \otimes P \otimes D \\ & \downarrow & & \downarrow \\ (C \otimes V) \otimes N(m) \longrightarrow (C \otimes V) \otimes Q \otimes D. \end{array}$$

Since the right vertical map is a fibration in  ${}_{C}\mathsf{CoMod}_{D}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ , then so is the left vertical map. Thus  $\{(C \otimes V) \otimes Y(m)\}$  is a fibrant tower of (C, D)-bicomodules.

**Lemma A.9.** Let C and D be simply connected dg-coalgebras over  $\Bbbk$ . If M is a fibrant left C-comodule and N is a fibrant right D-comodule, then  $M \otimes N$  is a fibrant (C, D)-bicomodule.

*Proof.* Let  $\{M(n)\}$  be the Postnikov tower in  ${}_{C}\mathsf{CoMod}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$  of X. As M is fibrant, it is a retract of  $\widetilde{X} = \lim_{n \to \infty} M(n)$ . Therefore it is sufficient to show that  $\widetilde{M} \otimes N$  is a fibrant (C, D)-bicomodule. Since the tower stabilizes in each degree, we have

$$(\lim_{n \to \infty} M(n)) \otimes N \cong \lim_{n \to \infty} (M(n) \otimes N),$$

and thus it is enough to show that  $\{M(n) \otimes N\}$  is a fibrant tower of (C, D)-bicomodules. For n = 0, we have  $M(0) \otimes N = 0$ , which is fibrant. For n = 1, we have  $M(1) \otimes N = (C \otimes M) \otimes N$ , which is a fibrant (C, D)-bicomodule by Lemma A.8. For  $n \ge 1$ , we obtain the pullback in  ${}_{C}\mathsf{CoMod}_{D}(\mathsf{Ch}_{\Bbbk}^{\ge 0})$ :

$$M(n+1) \otimes N \longrightarrow (C \otimes P) \otimes N$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$M(n) \otimes N \longrightarrow (C \otimes Q) \otimes N.$$

The right vertical map is a fibration in  ${}_{C}\mathsf{CoMod}_{D}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$  by Lemma A.6 since its kernel  $(C \otimes K) \otimes N$  is a fibrant (C, D)-bicomodule by Lemma A.8, where K is the kernel of  $P \to Q$ . Thus the left vertical map is a fibration.

**Proposition A.10.** Let C, D and E be simply connected dg-coalgebras over  $\Bbbk$ . If M is a fibrant (D, C)-bicomodule and N is a fibrant (C, E)-bicomodule, then  $M \Box_C N$  is a fibrant (D, E)-bicomodule.

Proof. Let  $\{M(n)\}$  be the Postnikov tower of M in  ${}_{D}\mathsf{CoMod}_{C}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ . Then M is a retract of  $\widetilde{M} \simeq \lim_{n} M(n)$ . As  $\widetilde{M} \square_{C} N = (\lim_{n} M(n) \square_{C} N) \cong \lim_{n} (M(n) \square_{C} N)$ , it is enough to show that  $\{M(n) \square_{C} N\}$  is a fibrant tower of (D, E)-bicomodules. For n = 0, then  $M(0) \square_{C} N = 0$  and is thus fibrant. For n = 1, we get  $M(1) \square_{C} N \cong (D \otimes M \otimes C) \square_{C} N \cong (D \otimes M) \otimes N$ . By Lemmas A.7 and A.9, it is a fibrant (D, E)-bicomodule. As the functor  $-\square_{C} Y$  preserves pullbacks, we obtain the pullback in  ${}_{D}\mathsf{CoMod}_{E}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ :

$$\begin{array}{ccc} M(n+1) \square_C N & \longrightarrow & (D \otimes P) \otimes N \\ & & \downarrow & & \downarrow \\ M(n) \square_C N & \longrightarrow & (D \otimes Q) \otimes N. \end{array}$$

The right vertical map is a fibration by Lemma A.6 as its kernel is  $(D \otimes K) \otimes N$ , which is a fibrant (D, E)bicomodule by Lemmas A.7 and A.9, where K is the kernel of  $P \to Q$ . Thus the left vertical map is a fibration.

One particularly nice characterizing algebraic property of fibrant comodules is that they are *coflat*, i.e., they interplay well with the cotensor product and exact sequences. The cotensor product is left-exact, as it preserves finite products and equalizers. We are interested in knowing the cases in which it preserves exactness, without any cocommutativity requirement.

**Definition A.11.** Let C and D be flat coalgebras in a symmetric monoidal abelian category,  $(C, \otimes, \mathbb{I})$ . Let M be a (C, D)-bicomodule. We say M is left coflat over C if given any short exact sequence in  $CoMod_C$ 

$$0 \longrightarrow N \longrightarrow N' \longrightarrow N'' \longrightarrow 0,$$

we obtain a short exact sequence in  $\mathsf{C}$ 

$$0 \longrightarrow N \square_C M \longrightarrow N' \square_C M \longrightarrow N'' \square_C M \longrightarrow 0.$$

Similarly, we say M is right coflat over D if given any short exact sequence in <sub>D</sub>CoMod

 $0 \longrightarrow N \longrightarrow N' \longrightarrow N'' \longrightarrow 0,$ 

we obtain a short exact sequence in  $\mathsf{C}$ 

$$0 \longrightarrow M \square_D N \longrightarrow M \square_D N' \longrightarrow M \square_D N'' \longrightarrow 0$$

We say the bicomodule M is two-sided coflat over (C, D) if it is left coflat over C and right coflat over D. Using Definition 1.13, if M is right coflat over C, or if N is left coflat over C, then  $\mathsf{CoTor}^i_C(M, N) = 0$  for all  $i \geq 1$ .

**Proposition A.12.** Let C and D be simply connected dg-coalgebras over  $\Bbbk$ . If a (C, D)-bicomodule is fibrant, then it is a two-sided coflat bicomodule over (C, D).

*Proof.* Notice that any cofree (C, D)-bicomodule  $C \otimes V \otimes D$  is two-sided coflat. Let us show that coflatness is preserved under extensions. Consider a short exact sequence in  $_C CoMod_D(Ch_{\Bbbk}^{\geq 0})$ ,

 $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0.$ 

Suppose M and P are two-sided coflat. Let Q be a left D-comodule. We then obtain an exact sequence

$$\mathsf{CoTor}_1^D(M,Q) \longrightarrow \mathsf{CoTor}_1^D(N,Q) \longrightarrow \mathsf{CoTor}_1^D(P,Q).$$

By exactness, we get that  $\mathsf{CoTor}_1^D(N,Q) = 0$  for any left *D*-comodule *Q*. Thus *N* is coflat as a right *D*-comodule. We argue similarly to show that *N* is coflat as a left *C*-comodule.

Now let F be a fibrant (C, D)-bicomodule. Let  $\{F(n)\}$  be its Postnikov tower in  ${}_{C}\mathsf{CoMod}_{D}(\mathsf{Ch}_{\Bbbk}^{\geq 0})$  and write  $\widetilde{F} = \lim_{n \to \infty} F(n)$ . Since F is a retract of  $\widetilde{F}$ , then for any left D-comodule Q, we get  $\mathsf{CoTor}_{1}^{D}(F, Q)$  is a retract of  $\mathsf{CoTor}_{1}^{D}(\widetilde{F}, Q)$ . Thus if  $\widetilde{F}$  is coflat as a right D-comodule, so is F. We prove  $\widetilde{F}$  is right coflat by

induction. For n = 0, we see that F(0) = 0 is coflat. For n = 1, we get that  $F(1) = C \otimes F \otimes D$ , a cofree bicomodule, and is thus right coflat. Suppose we have shown that F(n) is right coflat over D. Then from the short exact sequence in  $_{C}CoMod_{D}(Ch_{\Bbbk}^{\geq 0})$ :

$$0 \longrightarrow C \otimes K \otimes D \longrightarrow F(n+1) \longrightarrow F(n) \longrightarrow 0,$$

we get that F(n+1) is also right coffat over D. Given any short exact sequence in  $_D$ CoMod(Ch<sup> $\geq 0$ </sup><sub>k</sub>):

 $0 \longrightarrow Q \longrightarrow Q' \longrightarrow Q'' \longrightarrow 0,$ 

we obtain a short exact sequence of towers in  $Ch_{k}^{\geq 0}$ :

$$0 \longrightarrow \{F(n) \Box_D W\} \longrightarrow \{F(n) \Box_D W'\} \longrightarrow \{F(n) \Box_D W''\} \longrightarrow 0.$$

Each of these towers satisfies the Mittag-Leffler condition, and since limit of towers that stabilizes in each degree commute with cotensor product, we obtain a short exact sequence

$$0 \longrightarrow \widetilde{F} \square_D Q \longrightarrow \widetilde{F} \square_D Q' \longrightarrow \widetilde{F} \square_D Q'' \longrightarrow 0.$$

Therefore  $\widetilde{F}$  is right coflat over D. We can show  $\widetilde{F}$  is left coflat over C in a similar fashion.

**Remark A.13.** In fact, the result of [Pér20, 4.16] remains true in the non-commutative case. Thus one can show that a (C, D)-bicomodule is fibrant if and only if it is coflat as a right  $(C^{op} \otimes D)$ -comodule. We shall not need this result here so we do not provide details, however we do mention below the relationship between two-sided coflat (C, D)-comodules and right coflat  $(C^{op} \otimes D)$ -comodules.

**Lemma A.14.** Let  $(C, \otimes, \mathbb{I})$  be a symmetric monoidal category. Let  $(C, \Delta_C, \varepsilon_C)$  and  $(D, \Delta_D, \varepsilon_D)$  be flat coalgebras in C. Let M be a right  $(C \otimes D)$ -comodule. Let N be a left D-comodule. Then  $C \otimes N$  is a left  $(C \otimes D)$ -comodule, and we obtain an isomorphism in C:

$$M\square_{C\otimes D}(C\otimes N)\cong M\square_D N.$$

*Proof.* Let  $\lambda: N \to D \otimes N$  be the left *D*-coaction on *N*. Then we obtain a left  $(C \otimes D)$ -coaction on *N* via

$$C\otimes N \xrightarrow{\mathrm{id}_C\otimes\lambda} C\otimes D\otimes N \xrightarrow{\Delta_C\otimes\mathrm{id}_{D\otimes Y}} C\otimes C\otimes D\otimes N\cong C\otimes D\otimes C\otimes N.$$

The above coaction, denoted  $\lambda : C \otimes N \to (C \otimes D) \otimes C \otimes N$ , is counital and coassociative because the left *D*-coaction on *N* and the coalgebra structure on *C* are both coassociative and counital.

To prove the desired isomorphism, we verify that  $M \Box_D N$  satisfies the universal property of the equalizer. From the right  $(C \otimes D)$ -coaction  $\rho : M \to M \otimes C \otimes D$ , we obtain the underlying right C-coaction  $\rho_C$  on M as the composite

$$M \xrightarrow{\rho} M \otimes C \otimes D \xrightarrow{\operatorname{id}_{X \otimes C} \otimes \varepsilon_D} M \otimes C$$

Now we obtain a morphism  $\alpha: M \square_D N \to M \square_{C \otimes D} (C \otimes N)$  by functoriality of the equalizers.

$$\begin{array}{cccc} M \Box_D N & & & M \otimes N & & & & & & & \\ & & & & & & & & \\ & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & &$$

The right unlabeled vertical arrow is induced by applying the functor  $-\otimes N$  on the map

$$M \otimes D \xrightarrow{\rho_C \otimes \mathrm{id}_D} (M \otimes C) \otimes D \xrightarrow{\mathrm{id}_M \otimes \Delta_C \otimes \mathrm{id}_D} \underbrace{(M \otimes (C \otimes C)) \otimes D}_{\cong M \otimes (C \otimes D) \otimes C}.$$

Similarly, by applying the counit map  $\varepsilon_C : C \to \mathbb{I}$  vertically on each C, we obtain the dashed map on the equalizers below:

The induced morphism  $\beta$  is the inverse of the morphism  $\alpha$  defined above. Therefore we obtain the desired isomorphism in C.

**Lemma A.15.** Let  $(C, \otimes, \mathbb{I})$  be a symmetric monoidal abelian category. Let C and D be flat coalgebras in C. Let M be a (C, D)-bicomodule in C. If M is right coflat as a right  $(C^{op} \otimes D)$ -comodule, then it is two-sided coflat as a (C, D)-bicomodule.

*Proof.* Suppose M is right coflat a  $(C^{op} \otimes D)$ -comodule. Let us first show that M is right coflat over D. Consider a short exact sequence in  $_D$ CoMod:

$$0 \longrightarrow N \longrightarrow N' \longrightarrow N'' \longrightarrow 0.$$

By the previous lemma, we obtain that  $C^{\mathsf{op}} \otimes N$ ,  $C^{\mathsf{op}} \otimes N'$ , and  $C^{\mathsf{op}} \otimes N''$  are left  $(C^{\mathsf{op}} \otimes D)$ -comodules. Since  $C^{\mathsf{op}} \otimes -$  preserves exactness because it is flat, we obtain a short exact sequence in  $(C^{\mathsf{op}} \otimes D)$ -CoMod:

 $0 \longrightarrow C^{\mathsf{op}} \otimes N \longrightarrow C^{\mathsf{op}} \otimes N' \longrightarrow C^{\mathsf{op}} \otimes N'' \longrightarrow 0.$ 

Since M is a right coflat over  $(C^{op} \otimes D)$ , we obtain that the sequence is exact in C:

$$0 \longrightarrow M \square_{C^{\mathsf{op}} \otimes D} C^{\mathsf{op}} \otimes N \longrightarrow M \square_{C^{\mathsf{op}} \otimes D} C^{\mathsf{op}} \otimes N' \longrightarrow M \square_{C^{\mathsf{op}} \otimes D} C^{\mathsf{op}} \otimes N'' \longrightarrow 0$$

By the previous lemma, this induces the short exact sequence in C:

$$0 \longrightarrow M \square_D N \longrightarrow M \square_D N' \longrightarrow M \square_D N'' \longrightarrow 0.$$

Thus M is right coflat over D as desired. We prove that M is left coflat over C in a similar fashion.  $\Box$ 

**Example A.16.** The converse is not true: a two-sided coflat (C, D)-bicomodule need not to be a right coflat  $(C^{op} \otimes D)$ -comodule, and thus need not be a fibrant (C, D)-bicomodule. This can already be seen for modules over an algebra (even commutative). Indeed, consider the polynomial ring,  $\Bbbk[x]$ , as a  $\Bbbk$ -algebra. It is a two-sided flat  $\Bbbk[x]$ -module, but it is not flat as a right  $(\Bbbk[x] \otimes \Bbbk[x])$ -module. Indeed, note first that  $\Bbbk[x, y] \cong \Bbbk[x] \otimes \Bbbk[x]$ . Consider the following short exact sequence of left k[x, y]-modules

 $0 \longrightarrow \Bbbk[x,y] \stackrel{\cdot y}{\longrightarrow} \Bbbk[x,y] \longrightarrow \Bbbk[x] \longrightarrow 0.$ 

If we apply  $\Bbbk[x] \otimes_{\Bbbk[x,y]} -$ , we obtain the exact sequence

$$\Bbbk[x] \longrightarrow \Bbbk[x] \longrightarrow \Bbbk[x] \longrightarrow 0.$$

The above sequence is not a short exact sequence however, which shows that  $\mathbb{k}[x]$  is not flat as a right  $\mathbb{k}[x, y]$ -module.

We are now equipped to prove a crucial result in this paper: the cotensor product with a fibrant comodule preserves quasi-isomorphisms.

**Proposition A.17.** Let C and D be simply connected dg-coalgebras over  $\Bbbk$ . Let M be a fibrant (C, D)-bicomodule. Then  $M \square_D - : {}_D \mathsf{CoMod}(\mathsf{Ch}_{\Bbbk}^{\geq 0}) \to \mathsf{Ch}_{\Bbbk}^{\geq 0}$  and  $-\square_C M : \mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0}) \to \mathsf{Ch}_{\Bbbk}^{\geq 0}$  preserve weak equivalences.

*Proof.* We use an Eilenberg-Moore spectral sequence argument. Let N be any left D-comodule in  $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ . As in Remark 3.11, the conormalized cobar complex  $\underline{\Omega}^{\bullet}(M, D, N)$  is a second quadrant double chain complex that is bounded. We grade the row cohomologically, but the columns homologically. Hence its two associated spectral sequences converge to the same page, see [McC01, 2.15].

The first spectral sequence has its  $E^1$ -page induced by the cohomology of the rows and therefore

$$E^1_{\bullet,q} = H^q(\underline{\Omega}^{\bullet}(M, D, N)) \cong \mathsf{CoTor}^q_D(M, N).$$

Since M is a fibrant (C, D)-bicomodule, then it is coflat as a right D-comodule. Thus  $E_{\bullet,q}^1 = 0$  for all  $q \ge 1$ , and  $E_{\bullet,0}^1 = M \Box_D N$ . Hence the spectral sequence collapses onto its second page,  $E_{\bullet,0}^2 = H_*(M \Box_D N)$ .

The second spectral sequence has its  $E^1$ -page induced by the homology of its columns and therefore

$$E_{\bullet,q}^{1} = H_{*}(\underline{\Omega}^{q}(X, D, Y)) = \underline{\Omega}^{q}(H_{*}(M), H_{*}(D), H_{*}(N)).$$

Thus, as its  $E^2$ -page is given by the cohomology of the induced cochain complex, we obtain

$$E^{2}_{\bullet,q} = \text{CoTor}^{q}_{H_{*}(D)}(H_{*}(M), H_{*}(N)).$$

It converges to the page with trivial columns except its  $0^{th}$  column, which is given by the cohomology  $H_*(M \square_D N)$ .

Combining the arguments above, we obtain a converging Eilenberg-Moore spectral sequence

$$E^2_{\bullet,q} = \mathsf{CoTor}^q_{H_*(D)}(H_*(M), H_*(N)) \Rightarrow E^\infty_{\bullet,0} = H_*(M \Box_D N),$$

for any left D-comodule N. In particular, given a map of left D-comodules  $N \to N'$  that is a quasiisomorphism, we get

$$\operatorname{CoTor}_{H_*(D)}^q(H_*(M), H_*(N)) \cong \operatorname{CoTor}_{H_*(D)}^q(H_*(M), H_*(N')).$$

Thus the map  $N \to N'$  induces an isomorphism,  $H_*(M \square_D N) \cong H_*(M \square_D N')$ .

**Corollary A.18.** Let C, D and E be simply connected coalgebras in  $Ch_{\Bbbk}^{\geq 0}$ . Let M be a fibrant (D, C)-bicomodule and N a fibrant (C, E)-bicomodule. Then we obtain a quasi-isomorphism of (D, E)-bicomodules

$$M \square_C N \simeq \Omega(M, C, N).$$

**Corollary A.19.** Let C, D, and E be simply connected coalgebras in  $Ch_{\mathbb{k}}^{\geq 0}$ . Let M be a fibrant (C, D)-bicomodule, let N be a left fibrant D-comodule and P be a fibrant right E-comodule. Then there is a natural quasi-isomorphism of (C, E)-bicomodules

$$\Omega(M, D, N) \otimes P \simeq \Omega(M, C, N \otimes P).$$

*Proof.* Since M and N are fibrant, we obtain

$$\Omega(M, D, N) \otimes P \simeq (M \Box_D N) \otimes P$$
$$\cong M \Box_D (N \otimes P)$$
$$\simeq \Omega(M, D, N \otimes P).$$

The isomorphism follows from the fact that P is flat in  $Ch_{\Bbbk}^{\geq 0}$ , and thus preserves equalizers. The last quasiisomorphism follows from the fact that  $N \otimes P$  remains a fibrant (D, E)-bicomodule by Lemma A.9.

**Remark A.20.** Corollary A.19 is not true if the comodules are not fibrant, since in general the tensor product does not preserve infinite homotopy limits.

**Corollary A.21.** Let C, D, E, and F be simply connected coalgebras in  $Ch_{\mathbb{k}}^{\geq 0}$ . Let M be a fibrant (C, D)-bicomodule, let N be a fibrant (D, E)-bicomodule, and let P be a fibrant (E, F)-bicomodule. Then we obtain a quasi-isomorphism of (C, F)-bicomodules

$$\Omega(\Omega(M, D, N), E, P) \simeq \operatorname{holim}_{\Delta \times \Delta} \Omega^{\bullet}(\Omega^{\bullet}(M, D, N), E, P).$$

*Proof.* By the previous Corollary, for any  $i \ge 0$ 

$$\operatorname{holim}_{\Delta}(\Omega^{\bullet}(M, D, N) \otimes E^{\otimes i} \otimes P) \simeq \operatorname{holim}_{\Delta}(\Omega^{\bullet}(M, D, N \otimes E^{\otimes i} \otimes P))$$
$$\simeq \Omega(M, D, N \otimes E^{\otimes i} \otimes P)$$
$$\simeq \Omega(M, D, N) \otimes E^{\otimes i} \otimes P$$
$$\simeq \operatorname{holim}_{\Delta}(\Omega^{\bullet}(M, D, N)) \otimes E^{\otimes i} \otimes P.$$

In other words, we have proved that we obtain a quasi-isomorphism between fibrant (C, F)-bicomodules (recall that we fixed a model for the homotopy limit):

$$\operatorname{holim}_{\Delta}(\Omega^{i}(\Omega^{\bullet}(M, D, N), E, P)) \simeq \Omega^{i}(\operatorname{holim}_{\Delta}\Omega^{\bullet}(M, D, N), E, P).$$

This above equivalence is compatible with cofaces and codegeneracies of the cobar construction, and therefore we obtain by [Hir03, 18.5.2, 18.5.3]:

$$\Omega(\Omega(M, D, N), E, P) \simeq \operatorname{holim}_{\Delta} \Omega^{\bullet} \left( \Omega(M, D, N), E, P \right)$$
  
$$\simeq \operatorname{holim}_{\Delta} \Omega^{\bullet} \left( \operatorname{holim}_{\Delta} \Omega^{\bullet} (M, D, N), E, P \right)$$
  
$$\simeq \operatorname{holim}_{\Delta} \operatorname{holim}_{\Delta} \Omega^{\bullet} \left( \Omega^{\bullet}(M, D, N), E, P \right)$$
  
$$\simeq \operatorname{holim}_{\Delta \times \Delta} \Omega^{\bullet} (\Omega^{\bullet}(M, D, N), E, P).$$

**Corollary A.22.** Let C, D, and E be simply connected coalgebras in  $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ . Let M be a fibrant (C, D)-bicomodule and N a fibrant (D, E)-bicomodule. Then we obtain a quasi-isomorphism

 $\operatorname{coHH}(\Omega(M, D, N), C) \simeq \operatorname{holim}_{\Delta \times \Delta} \operatorname{coHH}^{\bullet}(\Omega^{\bullet}(M, D, N), E, P).$ 

*Proof.* By Corollary A.19, for any i > 0

$$\operatorname{holim}_{\Delta}(\Omega^{\bullet}(M, D, N) \otimes C^{\otimes i}) \simeq \operatorname{holim}_{\Delta}(\Omega^{\bullet}(M, D, N)) \otimes C^{\otimes i}$$

The above equivalence is compatible with cofaces and codegeneracies of the cobar construction, and therefore we obtain [Hir03, 18.5.2, 18.5.3]:

$$\operatorname{coHH}(\Omega(M, D, N), C) \simeq \operatorname{holim}_{\Delta} \operatorname{coHH}^{\bullet}(\operatorname{holim}_{\Delta} \Omega^{\bullet}(M, D, N), C)$$
$$\simeq \operatorname{holim}_{\Delta \times \Delta} \operatorname{coHH}^{\bullet}(\Omega^{\bullet}(M, D, N), C).$$

Appendix B. CoHochschild homology in higher categories

In [HR21b], the authors give an  $\infty$ -categorical definition of topological Hochschild homology with coefficients, extending the approach of [NS18]. In [BP23], a dual approach of [NS18] for topological coHochschild homology was given. We provide here a dual approach of [HR21b].

Let  $\mathcal{C}$  be a symmetric monoidal  $\infty$ -category. Denote by  $\mathcal{AB}$  the 2-colored operad whose algebras are pairs (A, M) consisting of an associative algebra A and a bimodule M over A. Denote  $\mathcal{AB}^{\otimes}$  its operator category as in [Lur17, 2.1.1.7]. Therefore objects in  $\operatorname{Alg}_{\mathcal{AB}}(\mathcal{C})$  are pairs (A, M) where A is an  $\mathbb{A}_{\infty}$ -algebra in  $\mathcal{C}$  and M is an (A, A)-bimodule in  $\mathcal{C}$ .

Let us denote the underlying  $\infty$ -category of the symmetric monoidal envelope of an  $\infty$ -operad  $\mathcal{O}^{\otimes}$  by  $Env(\mathcal{O}^{\otimes})$ , as in [Lur17, 2.2.4.3]. Given (A, M) in  $Alg(\mathcal{C})$ , the authors in [HR21b] induce a map

 $(A, M) : \operatorname{Env}(\mathcal{N}(\mathcal{AB}^{\otimes})) \longrightarrow \operatorname{Env}(\mathcal{C}^{\otimes}),$ 

which assembles into a functor

$$\mathcal{N}(\Delta^{\mathsf{op}}) \longrightarrow \mathsf{Env}(\mathcal{N}(\mathcal{AB}^{\otimes})) \xrightarrow{(A,M)} \mathsf{Env}(\mathcal{C}^{\otimes}) \xrightarrow{\otimes} \mathcal{C}.$$

The first map is a lift of the simplicial circle  $S^1 : \Delta^{\mathsf{op}} \to \mathsf{Fin}$  over the coCartesian fibration  $\mathsf{Env}(\mathcal{N}(\mathcal{AB}^{\otimes})) \to \mathsf{Fin}$ Fin. Essentially, the object [n] in  $\Delta^{op}$  is sent to  $A^{\otimes n} \otimes M$ . The above is making precise the cyclic bar construction  $\mathsf{CBar}^{\mathcal{C}}_{\bullet}(A, M)$ :

$$\cdots \Longrightarrow A \otimes A \otimes M \Longrightarrow A \otimes M \Longrightarrow M.$$

We therefore obtain a functor

$$\mathsf{CBar}^{\mathcal{C}}_{\bullet}(-,-):\mathsf{Alg}_{\mathcal{AB}}(\mathcal{C})\longrightarrow\mathsf{Fun}\Big(\mathcal{N}(\Delta^{\mathsf{op}}),\mathcal{C}\Big).$$

Applying this construction to  $\mathcal{C}^{op}$  then yields

$$\mathsf{CBar}^{\mathcal{C}^{\mathsf{op}}}_{\bullet}(-,-):\mathsf{Alg}_{\mathcal{AB}}(\mathcal{C}^{\mathsf{op}})\longrightarrow\mathsf{Fun}\Big(\mathcal{N}(\Delta^{\mathsf{op}}),\mathcal{C}^{\mathsf{op}}\Big).$$

Taking the opposite functor leads to

$$\mathsf{CoAlg}_{\mathcal{AB}}(\mathcal{C}) = (\mathsf{Alg}_{\mathcal{AB}}(\mathcal{C}^{\mathsf{op}}))^{\mathsf{op}} \to \mathsf{Fun}\Big(\mathcal{N}(\Delta^{\mathsf{op}}), \mathcal{C}^{\mathsf{op}}\Big)^{\mathsf{op}} \simeq \mathsf{Fun}\Big(\mathcal{N}(\Delta), \mathcal{C}\Big),$$

defining the cyclic cobar construction  $\mathsf{CoBar}^{\bullet}_{\mathcal{C}}(-,-)$ :  $\mathsf{CoAlg}_{\mathcal{AB}}(\mathcal{C}) \to \mathsf{Fun}(\mathcal{N}(\Delta),\mathcal{C})$ . Essentially, given an  $\mathbb{A}_{\infty}$ -coalgebra C and a (C,C)-bicomodule M in  $\mathcal{C}$ , the construction  $\mathsf{CoBar}^{\bullet}_{\mathcal{C}}(C,M)$  is making precise the diagram

$$\cdots \overleftarrow{\longleftarrow} C \otimes C \otimes M \overleftarrow{\longleftarrow} C \otimes M \overleftarrow{\longleftarrow} M.$$

**Definition B.1.** Let  $\mathcal{C}$  be a symmetric monoidal  $\infty$ -category that admits totalizations. Let C be an  $\mathbb{A}_{\infty}$ coalgebra in  $\mathcal{C}$  and let M be a (C, C)-bicomodule in  $\mathcal{C}$ . The coHochschild homology of C with coefficients in M, denoted coHH<sup> $\mathcal{C}$ </sup>(C, M), is defined to be the totalization in  $\mathcal{C}$  of the cosimplicial object obtained by the cyclic cobar construction CoBar $^{\mathcal{C}}_{\mathcal{C}}(C, M)$ . Per usual, we denote coHH<sup> $\mathcal{C}$ </sup>(C, C) by coHH<sup> $\mathcal{C}$ </sup>(C).

Recall from Definition 4.6 that, given a model category M with a monoidal structure, the coHochschild homology  $\operatorname{coHH}^{\mathsf{M}}(M, C)$  of a (strictly coassociative and counital) bicomodule M over a (strictly coassociative and counital) coalgebra C in M is the homotopy limit of the cosimplicial object given by the cyclic cobar complex in M as defined above.

If M is a combinatorial symmetric monoidal model category with class of weak equivalences denoted by W, then one could also obtain a symmetric monoidal  $\infty$ -category  $\mathcal{N}(M_c)[W^{-1}]$  obtained by Dwyer-Kan localization. Then one could consider coHH<sup> $\mathcal{N}(M_c)[W^{-1}]$ </sup> (M, C), as in Definition B.1.

**Proposition B.2.** Let M be a combinatorial symmetric monoidal model category. Let C be a coassociative and counital coalgebra in M, and let M be a (C, C)-bicomodule. Suppose both C and M are cofibrant in M. Then we obtain a weak equivalence

$$\operatorname{coHH}^{\mathsf{M}}(M, C) \simeq \operatorname{coHH}^{\mathcal{N}(\mathsf{M}_c)[\mathsf{W}^{-1}]}(M, C).$$

Proof. The localization functor  $\mathcal{N}(M_c) \to \mathcal{N}(M_c)[W^{-1}]$  is strong monoidal [Lur17, 1.3.4.25]. Therefore the cyclic bar cosimplicial object in Definition 4.6 carries over  $\mathcal{N}(M_c)[W^{-1}]$ . Thus by [Lur17, 1.3.4.23], the homotopy limit in M corresponds to the limit in the  $\infty$ -category  $\mathcal{N}(M_c)[W^{-1}]$ .

**Example B.3.** Our main example of interest is  $M = Ch_{\mathbb{k}}^{\geq 0}$ . In this case, when *C* is simply connected, the definition of coHochschild homology coincides with [HPS09], see also Remark 3.11 and [BGH<sup>+</sup>18, 2.11].

In what follows, we write  $\operatorname{coHH}^{\mathsf{Ch}_{\Bbbk}^{\geq 0}}$  as coHH. We now reinterpret coHochschild homology as a derived cotensor product.

**Proposition B.4.** Let C be a simply connected dg-coalgebra over  $\Bbbk$ . Let M be a fibrant (C, C)-bicomodule. There is a quasi-isomorphism

$$\operatorname{coHH}(M, C) \simeq \Omega(M, C \otimes C^{\mathsf{op}}, C).$$

*Proof.* We denote the enveloping coalgebra by  $C^e = C \otimes C^{op}$ . Recall that we have the quasi-isomorphism  $C \simeq \Omega(C, C, C)$  as (C, C)-bicomodules. Thus we obtain

$$M\widehat{\Box}_{C^e}C \simeq M\widehat{\Box}_{C^e}\Omega(C,C,C).$$

Notice that each object in the cosimplicial diagram  $\Omega^{\bullet}(C, C, C)$  is a fibrant (C, C)-bicomodule by Lemma A.8. Thus  $\Omega(C, C, C)$  is a fibrant (C, C)-bicomodule by [Hir03, 18.5.2]. Therefore

$$M\widehat{\Box}_{C^e}\Omega(C,C,C) \simeq M \Box_{C^e}\Omega(C,C,C).$$

Since the functor  $M \square_{C^e}$  – preserves towers that stabilize in each degree, we get

$$M \square_{C^e} \Omega(C, C, C) \simeq \operatorname{holim} \left( M \square_{C^e} \Omega^{\bullet}(C, C, C) \right)$$

We have an isomorphism,  $\operatorname{coHH}^n(M, C) \cong M \square_{C^e} \Omega^n(C, C, C)$ , for each  $n \ge 0$ , induced by

$$M \otimes C^n \cong M \square_{C^e} (C^e \otimes C^n) \cong M \square_{C^e} (C \otimes C^n \otimes C),$$

given by the permutation of  $C^{op} \cong C$  past  $C^n$ . These isomorphisms are compatible with the cosimplicial structures and thus provide an isomorphism,  $\operatorname{coHH}^{\bullet}(M, C) \cong M \square_{C^e} \Omega^{\bullet}(C, C, C)$ .

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DEPARTMENT OF MATHEMATICAL AND COMPUTATIONAL SCIENCES, MARIAN UNIVERSITY, 3200 COLD SPRING ROAD, INDI-ANAPOLIS, IN, 46222, USA

Email address: sklanderman@marian.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, 209 SOUTH 33rd Street, Philadelphia, PA 19104, USA *Email address:* mperoux@sas.upenn.edu